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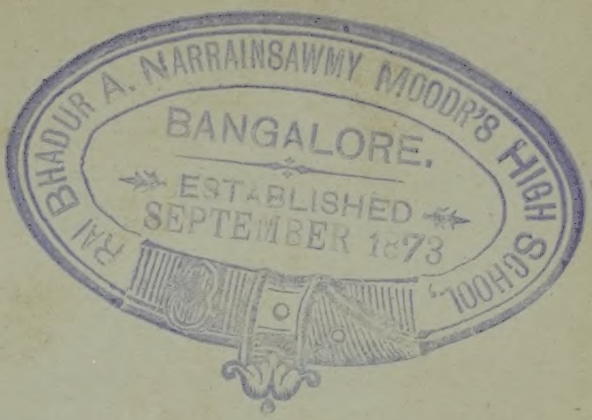

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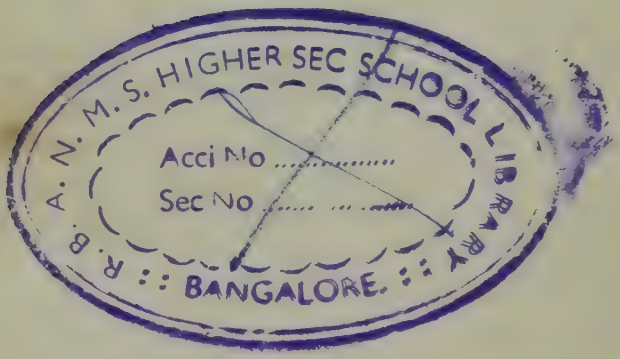

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PREFACE.

THIS Manual of Algebra has been specially prepared for the use of students who are reading for the Matriculation Examinations of the Indian Universities and will be found to embrace all the parts of the subject prescribed for any of these examinations, while it also includes all that is required for the School Final Examination of the University of Bombay and the Middle School Examination of the Punjab University.

The elementary principles are explained for the benefit of beginners more fully than is usual, and to ensure a sound and reliable ground-work all rules are traced to their origin. Thus no demand is made on the faith of a student unassisted by his intelligence, and he can examine for himself and verify the process by which each rule for conducting algebraical operations has been arrived at. Throughout the book preference has been given to the treatment of problems by inspection and formulae, which demand the intelligent appreciation of the student, rather than by the employment of mechanical and often only half comprehended rules. A great advance in mathematics will have been made by Indian students when once it has been impressed upon them that the application of what are called rules should be exceptional and not habitual, and that such rules as those for Division, H.C.F., L.C.M. and Evolution should be charily applied and only when methods by inspection fail.

In his treatment of Fractions the author has employed the method of the Master of Sidney Sussex College, Cambridge, by at once adopting a wide definition applicable to all fractions having numerator and denominator positive or negative, integral or fractional. In other parts of the book he has not consciously followed the guidance of any particular authority. He has advisedly omitted all consideration of negative or fractional indices, as being above the standard aimed at in the book or required by the Regulations of the Indian Universities in their Entrance Examinations.

Numerous original examples are appended to each chapter and have been constructed to bring into prominence points which the author's twenty-two years' experience in India as a mathematical teacher and examiner has shewn to be important, while many of them are by their local colouring specially adapted to Indian students. Answers to all the examples are supplied, together with frequent hints to assist the student in the solution of the examples on the first ten chapters. At the end of the book is a collection of problems selected from Examination Papers set in the Entrance and similar Examinations of the Universities of Bombay, Calcutta and Madras.

Every care has been taken to verify all examples and ensure accuracy in the answers, but it is feared that occasional errors may have escaped detection, and the author will be grateful for having his attention directed to them as well as for any hints which will enhance the educational value of the book.

J. T. HATHORNTHWAITE.

Elphinstone College, Bombay;
13*th September*, 1894.

CONTENTS.

CHAPTER		PAGE
I.	Definitions	1
II.	Addition	8
III.	Subtraction	16
IV.	Multiplication	21
V.	Division	32
VI.	Factors	45
VII.	Highest Common Factor	50
VIII.	Lowest Common Multiple	60
IX.	Fractions	66
X.	Involution, Evolution	85
XI.	Simple Equations	102
XII.	Problems which lead to Simple Equations	122
XIII.	Quadratic Equations	152
XIV.	Problems which lead to Quadratic Equations	169
XV.	Ratio, Proportion	180
	Miscellaneous Examples	192
	Answers	269

ALGEBRA.

CHAPTER I.

DEFINITIONS.

1. ALGEBRA is the science in which we employ letters to denote numbers and symbols to denote the operations performed upon them. The letters employed are the letters of some alphabet, generally of the English or the Greek, and the number of such letters may be increased at pleasure and to suit our convenience by appending figures or letters below the ordinary letters or by placing dashes above them. Thus we use

a, b, c ..... x, y, z ;
a', b', c' ..... x', y', z' ;
.....
a_1, b_1, c_1 ..... $x_1\ y_1, z_1$;
a_2, b_2, c_2 ..... x_2, y_2, z_2 ;
.....
a_m, b_m, c_m ..... x_m, y_m, z_m ;
α, β, γ .....

As a rule early letters of the alphabet a, b, c..... denote known or given quantities, and late letters of the alphabet x, y, z denote unknown quantities.

2. The symbol +, called the **plus** sign, denotes the operation of addition; thus $3 + 4$ or $4 + 3$ stands for the sum of 3 and 4 and is equivalent to 7; $9 + 23 + 58$ stands for the sum of 9, 23 and 58 and is equivalent to 90; $a + b$ or $b + a$ stands for the sum of a and b; and so on for any number of quantities which are to be added together, the order in which they are placed being a

matter of indifference. When any quantity such as a is to be added to itself, the result is not generally expressed by $a+a$ but by $2a$; and if a were again to be added, the result would not be expressed by $a+a+a$ but by $3a$, and so on. When a quantity has no sign prefixed the + sign is understood; thus a is equivalent to $+a$.

3. The symbol $-$, called the **minus** sign, denotes the operation of subtraction and is placed before the quantity which is to be subtracted from that which precedes it. Thus $4-3$ stands for the result of subtracting 3 from 4 and is equivalent to 1; $a-b$ stands for the result of subtracting b from a; $a-b-c$ stands for the result of subtracting c from $a-b$ and so on.

4. As in arithmetic the symbol $\times$ indicates that the quantities between which it is placed are to be multiplied together, and is often omitted for the sake of brevity; thus $a \times b$ or ab denotes the product of a and b. Similarly the symbol $\div$ indicates that the quantity which precedes it is to be divided by that which follows it; thus $a \div b$ denotes that a is to be divided by b. The operation of division is also frequently denoted by placing the dividend above the divisor with a line between them; thus $\frac{a}{b}$ expresses the same fact as $a \div b$.

5. The symbol $=$ denotes equality and indicates that the quantities between which it is placed are equal to each other. Thus $4+3=7$, $4-3=1$; $a+b=c$ states the fact that the sum of the quantities a and b is equal to the quantity c. Quantities connected by the sign of equality form what is called an **equation.**

6. The symbol $>$ denotes that the quantity preceding it is greater than that which follows it, and the symbol $<$ denotes that the quantity preceding it is less than that which follows it. Thus $4>3$ and $3<4$ state the same truth that 4 is greater than 3 and that 3 is less than 4; $a>b$ states that the quantity represented by a is greater than that represented by b.

7. In a train of reasoning the symbol $\because$ denotes 'since' or 'because,' and $\therefore$ denotes 'therefore'; thus $\because x=2$, $\therefore 3x=6$.

8. Any collection of algebraical symbols is called an **expression**; the parts of an expression which are connected by the

signs + and − are called **terms**. Thus $2a + 4b - 3c$ is an expression containing the terms $2a$, $4b$, $3c$. An expression consisting of one term is called a **simple** expression, and when it consists of more than one term it is called **compound**; thus $2a$ is a simple expression and $2a + 4b - 3c$ is compound. An expression consisting of two terms is called a **binomial**, of three terms a **trinomial** and of more than three terms a **multinomial**; thus $2a + 4b$ is a binomial, $2a + 4b - 3c$ is a trinomial, and $2a + 4b - 3c + d$ is a multinomial. Terms preceded by the sign + are called **positive** terms and preceded by the sign − are called **negative** terms; terms preceded by no sign have the sign + understood and are positive. Thus in the expression

$$a + b - c + d - e,$$

a, b and d are positive terms while c and e are negative.

9. When a quantity or expression consists of the product of other quantities or expressions, each of the latter is said to be a **factor** of the former; thus $a \times b \times c$ or abc is the product of a, b and c, and each of the quantities a, b, c is called a factor of abc. A product may consist of an arithmetical factor represented by a number and of another factor which is algebraical and represented by a letter or letters; in this case the numerical factor is called the **coefficient** of the other factor. Thus in the product $4abc$, 4 is the coefficient of abc. If a product contain zero as a factor, the product obviously vanishes. If a letter or product have no coefficient expressed, the coefficient 1 is understood, so that abc is equivalent to $1abc$.

10. When a quantity is multiplied by itself any number of times, each product so formed is called a **power** of that quantity and is expressed by placing above the symbol representing the quantity to the right the number which states how often the quantity is repeated in the product; and the figure or letter placed above a symbol to express the power of the symbol is called its **index** or **exponent**. Thus $a \times a$ or a^2 is the 2nd power or square of a, $a \times a \times a$ or a^3 is the 3rd power or cube of a, and so on. Also a^1 is the 1st power of a and is equivalent to a; and if a be continually multiplied by itself and occur n times in the product, the product is called the nth power of a and is denoted by a^n. Every power of 1 is 1 and every power of 0 is 0.

11. The **square root** of a quantity is that quantity which, multiplied by itself, produces the given quantity, and is denoted by the symbol $\surd$, called the **radical sign**, placed to the left of the given quantity, with the addition when required of a connecting line above it; thus the square root of 4 is denoted by $\surd 4$ and is 2, the square root of a^2 is denoted by $\surd a^2$ and is a, the square root of $a+b$ is expressed by $\surd\overline{a+b}$. The cube root of a quantity is that quantity which has the given quantity for its cube and is similarly denoted by the symbol $\sqrt[3]{}$; thus the cube root of 27 is denoted by $\sqrt[3]{27}$ and is 3, the cube root of a^3 is denoted by $\sqrt[3]{a^3}$ and is a, the cube root of $a+b$ is expressed by $\sqrt[3]{\overline{a+b}}$. In the same manner the fourth root of a quantity is that quantity which has the given quantity for its fourth power and is denoted by the symbol $\sqrt[4]{}$, and so on.

12. Each of the letters which occurs in an algebraical product is called a **dimension** of that product, and the number of letters is the **degree** of the product, a numerical coefficient being not reckoned. Thus $3abc$ or $3 \times a \times b \times c$, $4a^2b$ or $4 \times a \times a \times b$, $7a^3$ or $7 \times a \times a \times a$ are all of three dimensions and of the third degree. When all the terms of an algebraical expression are of the same degree it is said to be **homogeneous**; thus

$$4abc^2 + 3a^2bc - 7a^2b^2 + 11a^3b - 8a^4$$

is homogeneous, each of its terms being of the 4th degree.

The **degree of an expression** is the degree of the term of highest dimensions which occurs in it. Terms are said to be **like** when they consist of the same letters raised to the same powers, and are said to be **unlike** when this condition is not fulfilled; hence like terms differ only in their coefficients. Thus $5ab^2c$ and $8ab^2c$ are like while $7a^3b$ and $10a^2b^2$ are unlike. The terms of an expression that involve a common letter should as a matter of convenience be always arranged in ascending or descending powers of that letter, and should not be placed confusedly in any order the student chooses. Thus it is much better to write

$$8x^3 - 3ax^2 + 7a^2x + a^3$$

than $$7a^2x + 8x^3 + a^3 - 3ax^2$$

or $$a^3 - 3ax^2 + 8x^3 + 7a^2x.$$

13. Brackets ( ), { }, [ ] are employed to denote that two or more quantities enclosed within a pair of them are to be

treated as a single quantity. Thus $10+(3+4)$ means that the sum of 3 and 4, i.e. 7, is to be added to 10; $10-(4-3)$ means that the excess of 4 over 3, i.e. 1, is to be subtracted from 10; $10\times(3+4)$ means that 10 is to be multiplied by the sum of 3 and 4, i.e. by 7. Similarly $a+(b+c)$ means that the sum of b and c is to be added to a; $a-(b-c)$ means that the excess of b over c is to be subtracted from a; $a(b+c)$ means that a is to be multiplied by the sum of b and c; $(a+b)^3$ means that the sum of a and b is to be raised to the cube, and so on.

The effect of a bracket is also produced by drawing a straight line called a **vinculum** above the quantities which are to be treated as a single quantity; thus $a-\overline{b-c}$ is the same as $a-(b-c)$. The line mentioned in Art. 11 as sometimes placed above a quantity in connection with the radical sign is the vinculum.

14. Useful exercises in the use of the symbols which have been explained in this chapter are furnished by investigating the numerical values of algebraical expressions where the letters have certain assigned values.

Thus if $a=0$, $b=1$, $c=2$, $d=3$, let it be required to find the value of $\dfrac{a^d+d^c-b^c-c^d}{ad+cd-bc-ab}$ and of $\dfrac{(c^2-a^2)(d^2-b^2)}{(c+d-b)^2}$.

Here
$$\frac{a^d+d^c-b^c-c^d}{ad+cd-bc-ab}=\frac{0+9-1-8}{0+6-2-0}=\frac{0}{4}=0;$$

and
$$\frac{(c^2-a^2)(d^2-b^2)}{(c+d-b)^2}=\frac{(4-0)\times(9-1)}{(2+3-1)^2}$$
$$=\frac{4\times 8}{4^2}=\frac{32}{16}=2.$$

EXAMPLES.

If $a=2$, $b=4$, $c=1$, $d=5$, find the value of

1.	$5a$.	**2.**	$3c$.	**3.**	$2d$.	**4.**	b^2.	**5.**	$3c^3$.	**6.**	$5d^2$.
7.	ac.	**8.**	$7bd$.	**9.**	$10ad$.	**10.**	$2abc$.	**11.**	$12bc^2$.		
12.	$9d^3$.	**13.**	$20a^3b$.	**14.**	$3c^4$.	**15.**	$4abcd$.	**16.**	$8a^3d^2$.		
17.	$12b^2c^3$.	**18.**	$2d^5$.	**19.**	a^7.	**20.**	c^8.				

If $x=4$, $y=7$, $z=10$, find the value of

21.	$7x-4y$.	**22.**	$5x+10y-9z$.	**23.**	$3x+4y-4z$.	**24.**	$6x^2+2yz$.
25.	y^2-3x^2.	**26.**	$yz+zx+xy$.	**27.**	$xyz-4x^3$.	**28.**	$x^2y+z^2x-y^2z$.
29.	$z^3-x^3-y^3$.	**30.**	$4x^3z-y^4$.				

If $a=0$, $b=1$, $c=2$, $x=3$, $y=4$, $z=5$, evaluate:

31. $ax+by+cz$.
32. $5ax+3by-cz$.
33. $ayz+bzx+cxy$.
34. $b^2z-a^2y+c^2x$.
35. $(x+a)^2+y^2-z^2$.
36. $(x-a)^2-(y-b)^2$.
37. $2(x-a)^2-(y-b)^2-(z-c)^2$.
38. $2cx+3by-5az$.
39. $bcx+cay+abz$.
40. $a(y+z)+b(z+x)+c(x+y)$.
41. $x^2-4ax+b^2$.
42. $x^2-2bx-a^2$.
43. $2ax^2+6by^2-cz^2$.
44. $bxy-2cx-3ay+bc$.
45. $xyz+10cx+15by+12az-3yz-4zx-5xy$.

If $a=2$, $b=3$, $x=1$, $y=4$, evaluate:

46. $\frac{3a}{bx}$.
47. $\frac{6y}{ab}$.
48. $\frac{8ab}{3xy}$.
49. $\frac{a^2b^2}{x^2y^2}$.
50. $\frac{2abx}{3y}$.
51. $\frac{bxy}{2a}$.
52. $\frac{9axy}{4b}$.
53. $\frac{b^2x^2y^2}{a^4}$.
54. $\frac{b^3y^3}{9a^6}$.
55. $\frac{8a^3b}{x^3y^3}$.

If $a=1$, $b=2$, $c=0$, $d=3$, evaluate:

56. $\frac{3ab}{2d}$.
57. $\frac{bc}{ad}$.
58. $\frac{a^2c^2}{b^2d^2}$.
59. $\frac{5c^3d^4}{8a^5b^2}$.
60. $\frac{4a^3b^4c^5}{7d^8}$.

Evaluate the following when $a=0$, $b=1$, $c=6$, $d=9$, $e=12$:

61. $\frac{c+d}{a+e-b-c}$.
62. $\frac{2a+6b-c}{2a+6b+c}$.
63. $\frac{a+b+c+d+e}{b+e-c}$.
64. $\frac{9b+3e-5d}{9a+3c-5b}$.
65. $\frac{a+b+d-c}{a+c-2b}$.
66. $\frac{be+cd-ab}{e-b}$.
67. $\frac{a^2+b^2+e^2-c^2-d^2}{a+b+3d}$.
68. $\frac{a^2e^2}{cd}+\frac{2cd}{e}-\frac{b^2c^2}{4d}-\frac{dc}{3bc}$.
69. $\frac{3a^2+6b^2}{bc}+\frac{3c^2-d^2}{3d}-\frac{e^2-2cd}{3be}$.
70. $\frac{1}{a^2+b^2}-\frac{37}{b^2+c^2}+\frac{117}{c^2+d^2}-\frac{225}{d^2+e^2}$.
71. $\frac{d^2-1}{2d-8}-\frac{7d-3}{d+1}-\frac{11d+1}{5d+5}$.
72. $\frac{e^2+1}{2e+5}-\frac{7e-4}{e+4}+\frac{e-12}{e+3}$.
73. $\frac{ce}{c^2d^2-20b^2e^2}+\frac{b^2c^2-a^2d^2}{bd}$.
74. $\frac{ade+bce-acd-bcd}{a+6b+3c-2d+e}$.
75. $\frac{7e^2-a^3-b^3-c^3-d^3}{5d-2b-2c}$.

Evaluate the following when $x=3$, $y=2$, $z=0$:

76. $\frac{x^y-y^x}{x-y}$.
77. $\frac{x^y-z^x}{x-z}$.
78. $\frac{x^{2y}-y^{2x}}{x^y+y^x}$.
79. $\frac{\sqrt[3]{9x}-\sqrt[3]{4y}-\sqrt[3]{2z}}{\sqrt{3x}-\sqrt{2y}}$.
80. $\frac{\sqrt{6xy}-\sqrt{2yz}+\sqrt{5zx}}{\sqrt[4]{27x}-\sqrt[4]{8y}}$.

If $a=5$, $b=4$, $c=3$, $d=1$, find the value of

81. $(a+b)-(c+d)$. 82. $(a-b)+(c-d)$.

83. $a(b-c)+b(c-d)-c(a-d)-d(a-b)$. 84. $(a^2+d^2)-(b^2+c^2)$.

85. $(a+2d)^2-(b+c)^2$. 86. $(a-b)(c+d)-(a-d)(b-c)$.

87. $(a+b)(a-c)(a-d)-(a-b)(a+c)(a+d)$.

88. $(a+b-3c)(a-b+c)(a-c+d)$.

89. $(2a-b+3c)(3a-c+2d)(a-b-d)$.

90. $(a-b+d)^2+(a-c-d)^2+(b-c+d)^2-c^2$.

91. $(4acd-5bcd)-(abc-3abd)$. 92. $\sqrt{(a^2-c^2)}+\sqrt{(a^2-b^2)}-\sqrt{(b^2+c^2)}$.

93. $\sqrt{(bc+4d)}-\sqrt{(ab-4d)}$. 94. $\sqrt[3]{(b^3-b^2c+a^2b+c^2d)}-\sqrt[3]{(2bc+3d)}$.

95. $\sqrt[4]{(4a^2-6b^2+c^2-12d^2)}-\sqrt[4]{(a^2-b^2-c^2)}$.

96. Shew that $x^2-15x+56$ vanishes if $x=7$ or 8.

97. Shew that $x^3-14x^2+49x-36$ vanishes if $x=1$ or 4 or 9.

98. Indicate like terms in the expression

$$7a^2xy-4ab^2y-3axy^2+5a^2xy-6abx^2+10ab^2y.$$

99. Of how many dimensions are the terms $3a^2b^2c$, $7ab^3c^2$, $10abc^3$, a^3bc^2, $4a^2b^2c^2$, $11ab^2c^4$? Are any of them of the same degree?

100. Write down all the products of three dimensions that can be formed from a, b and c, and their powers.

CHAPTER II.

ADDITION.

15. In arithmetic all the quantities we meet with are positive, and we add positive quantities together or subtract a positive quantity from a positive quantity. But in algebra we have to deal with negative quantities, i.e. quantities preceded by the $-$ sign, as well as with positive quantities, i.e. quantities preceded by the $+$ sign expressed or understood. Hence addition in algebra has a sense it does not possess in arithmetic and frequently includes what we have been accustomed to regard as subtraction. The magnitude of a quantity, disregarding the sign, is called its *absolute* magnitude.

16. When the terms of an expression are connected by the $+$ sign, it is plainly a matter of no consequence in what order we place them, our only object being to collect them together and find the value of their aggregate. Thus $a+b+c$ is obviously the same as $b+c+a$ or $c+a+b$.

But when a negative quantity appears in an expression we have to consider what it means. A positive quantity by its nature implies an increase, and a negative quantity by its nature implies a decrease, having the exactly opposite effect to the corresponding positive quantity. Now the expression

$$-b+a$$

means that the negative quantity $-b$ is to be added to the positive quantity a, thus diminishing a to the same extent to which the addition of the corresponding positive quantity b would have increased a, so that

$$-b+a=a-b.$$

Thus $a-b$ and $-b+a$ have the same meaning, the order of the terms being a matter of indifference.

And when an expression contains several terms connected by the signs + and −, it is still of no consequence in what order we place them. For the expression

$$a+b-c$$

means that b is to be added to a and c subtracted from their sum, and

$$a-c+b$$

means that c is to be subtracted from a and b added to the remainder; and our arithmetical conceptions enable us at once to conclude that the result must be the same in the two cases. Hence

$$a+b-c=a-c+b.$$

Or this result may at once be obtained from the consideration that we have proved the order of the terms to be a matter of indifference in the case of two terms so that

$$b-c=-c+b;$$

hence, adding each of these equal quantities to a we have

$$a+b-c=a-c+b.$$

Also the order of the terms being a matter of indifference in the case of two terms, we have

$$a-c=-c+a;$$

so that the expression $a-c+b$ may be written

$$-c+a+b;$$

and the three expressions $a+b-c$, $a-c+b$ and $-c+a+b$ are all equivalent to each other.

Any other expression may be treated in the same manner, and we thus arrive at the conclusion that the order of the component terms of an algebraical expression is a matter of indifference.

17. We shall now prove that to add expressions together is the same as to add together the component terms of the expressions arranged in any order we please.

The meaning of $15+(3+2)$

is that the sum of 3 and 2 is to be added to 15, thus producing 20, and it is obviously a matter of indifference whether to 15 we add first 3 and then 2, or add the sum 3 + 2, i.e. 5, at once; hence

$$15+(3+2)=15+3+2.$$

In the same manner the meaning of the expression

$$a + (b + c)$$

is that the sum of b and c is to be added to a, and, as in the above numerical example, it is plainly immaterial whether to a we first add b and then c or the sum of b and c at once; hence

$$a + (b + c) = a + b + c.$$

Again the meaning of

$$15 + (3 - 2)$$

is that the excess of 3 over 2 is to be added to 15, and it is obviously immaterial whether we first add 3 to 15 and then subtract 2 from the sum, or subtract 2 from 3 and add the remainder to 15, the result being 16 in each case; hence

$$15 + (3 - 2) = 15 + 3 - 2.$$

Similarly the expression

$$a + (b - c)$$

means that the excess of b over c is to be added to a, and it is plainly immaterial whether we first add b to a and then subtract c from the sum, or subtract c from b and then add the remainder to a. For on adding b alone to a we have added too much by the quantity c, since we are required to add not b but b diminished by c, so that the sum $a + b$ must afterwards be diminished by c. Hence

$$a + (b - c) = a + b - c.$$

Similarly we may prove that

$$a + (b - c - d) = a + b - c - d$$

and so on; and this method of proof may be extended to compound expressions containing any number of terms.

We are thus led to the following rule regarding the use of brackets:

When an expression within brackets is preceded by the + sign, the brackets may be removed.

And conversely:

Any expression may be put within brackets and preceded by the + sign.

We have also shewn that to add expressions together is the same as to add together their component terms. And we have previously shewn that the terms of each expression may be arranged in any order we please, the order being a matter of indifference. It will therefore obviously be a convenient arrangement to collect into groups all like terms before we commence the operation of addition.

18. We are hence led to frame the following rules for addition in algebra.

When the terms to be added are like and of the same sign, their coefficients must be added, the common sign prefixed and the common letter or letters affixed. Thus

$$2a + 7a + 13a = 22a;$$

$$-3xy - 5xy - 10xy = -18xy.$$

When the terms to be added are like, but some positive and others negative, all the positive coefficients should be collected and formed into one sum, and all the negative coefficients should be collected and formed into another sum; the final sum is the difference of these two sums with the common letter or letters affixed, and has the sign of the greater sum. Thus

$$3a - 5a + 12a - 6a + 9a - 2a = 24a - 13a = 11a;$$

$$4xy - 7xy - 10xy + xy - 8xy + 11xy = 16xy - 25xy = -9xy.$$

When the terms to be added are not all like, those which are like must first be collected and formed into single terms, and the rest must be written down unchanged each preceded by its proper sign. Thus

$$5a + 3b - c + 7a - 8b + 10d - a + 2b + 2c = 11a - 3b + c + 10d.$$

19. The addition of compound expressions is however much facilitated by writing them down symmetrically in horizontal rows in such a way that like terms may be in vertical columns.

Thus let it be required to add together $7a - 4b + 3c$, $-10a + 3b - 7d$, $9a - 8b - c$, $-a + 5b + 6e$.

Arranging the expressions as suggested we have

$$\begin{array}{rrrr} 7a & -4b & +3c & \\ -10a & +3b & & -7d \\ 9a & -8b & -c & \\ -a & +5b & & +6e \\ \hline 5a & -4b & +2c & -7d+6e \end{array}$$

Similarly let it be required to find the sum of

$$x^3-5x^2+7x-2,\quad -7x^3+3x^2-6x+8,$$
$$2x^3-9x^2-10x-6,\quad 4x^3+12x^2+9x-1.$$

We have

$$\begin{array}{rrrr} x^3- & 5x^2+ & 7x- & 2 \\ -7x^3+ & 3x^2- & 6x+ & 8 \\ 2x^3- & 9x^2- & 10x- & 6 \\ 4x^3+ & 12x^2+ & 9x- & 1 \\ \hline & x^2 & & -1 \end{array}$$

In this example the sum of the coefficients of the positive terms involving x is exactly equal to the sum of the coefficients of the negative terms, and the terms in x are then said to *cancel each other*; likewise the terms in x^3 cancel each other, and so the sum contains no term in x or x^3.

The rules for fractional coefficients are the same as for integral coefficients. Thus let it be required to find the sum of $\frac{3}{5}a^2-\frac{2}{3}ab-\frac{1}{6}b^2$, $-\frac{7}{10}a^2+\frac{6}{7}ab-\frac{13}{22}b^2$, $\frac{2}{15}a^2-\frac{1}{21}ab+\frac{2}{3}b^2$. By the same method as for integral coefficients the sum is found to be $\frac{1}{30}a^2+\frac{1}{7}ab-\frac{1}{11}b^2$.

$$\begin{array}{rrr} \frac{3}{5}a^2- & \frac{2}{3}ab- & \frac{1}{6}b^2 \\ -\frac{7}{10}a^2+ & \frac{6}{7}ab- & \frac{13}{22}b^2 \\ \frac{2}{15}a^2- & \frac{1}{21}ab+ & \frac{2}{3}b^2 \\ \hline \frac{1}{30}a^2+ & \frac{1}{7}ab- & \frac{1}{11}b^2 \end{array}$$

20. Addition in algebra is performed by prefixing the + sign to the quantity to be added, and we know that to add b, i.e. $+b$, to a produces the result $a+b$, so that we have

$$+(+b)=+b.$$

Again by prefixing the + sign to the quantity $-b$ we add it to another quantity a and have seen that the result is $a-b$, so that

$$+(-b)=-b.$$

Hence *a positive sign operating on a positive sign produces a positive sign and operating on a negative sign produces a negative sign.*

We also notice that to add the negative quantity $-b$ is the same as to subtract the positive quantity b.

Such results as these expressed under the form of symbols are called formulæ. A **Formula** is a rule or general result expressed symbolically.

21. We have seen that negative quantities may be added to each other and that a negative quantity may be added to a positive quantity, and we shall subsequently see that a negative quantity may be subtracted from a positive quantity or from a negative quantity, and also that negative quantities obey the laws of multiplication and division.

In fact, apart from any idea of subtraction, which involves

only the magnitudes of the quantities with which we are dealing, negative quantities possess, inherent in themselves, a *quality* which distinguishes them from the corresponding positive quantities. This quality will be better understood by illustrations taken from ordinary life. Thus if a man possess ten rupees and is not in debt, his property would be described algebraically as 10 or $+10$; if he possess nothing but is not in debt, his property would be called zero; if he not only possess nothing but is ten rupees in debt, in algebraical language his property would be -10.

Again as a negative quantity is the reverse of the corresponding positive quantity and has exactly the opposite effect, if we agree to measure distances in one direction, say towards the north, by positive quantities, distances in the opposite direction, i.e. towards the south, would be represented by negative quantities. Thus if a man were to walk ten miles towards the north, algebraically he would be said to be at a distance 10 or $+10$ from the starting-point; and if he were to walk ten miles towards the south, he would be said to be at a distance -10. And if he walked first ten miles towards the north and then retracing his steps walked ten miles back towards the south, he would be at a distance $10-10$, i.e. 0, from his original starting-point, which means that he would have returned to it.

For another illustration, if we agree to measure time past a certain hour, as noon, by a positive quantity, then time before that hour would be expressed by the corresponding negative quantity. Thus if 1 p.m. were represented by 1, 11 a.m. might be called -1 and would be -1 p.m.

EXAMPLES.

Add together

1. $a-b,\ a+b.$ 2. $a-b,\ -a+b.$ 3. $2a+5b,\ 5a-7b,\ -6a+3b.$
4. $7a-2b,\ -10a+5b,\ 4a-3b.$ 5. $b+c-a,\ c+a-b,\ a+b-c.$
6. $a-2b+3c,\ 5a+7b-c,\ -6a-4b-2c.$
7. $3a+5b-2c,\ 7a-11b+d,\ 7b+c-4d.$
8. $a+2b-3c,\ 3b-5a+2c,\ 9c-8b-7a,\ 12a-7c+4b.$
9. $3a-5b+4c+7d,\ -7a+9b-11c-4d,$
   $13a-8b-2c-5d,\ -8a+2b+12c-2d.$

10. $2a^2-4ab+b^2$, $a^2+3ab-2b^2$, $-5a^2+7ab-3b^2$, $3a^2-6ab+4b^2$.

11. $a^2-5ac+ab-7bc-c^2$, $b^2+7ab-bc+3ac-a^2$,
$c^2+8bc+2ac-8ab-b^2$.

12. $2a^3+4a^2b-3ab^2-b^3$, $-5a^3-7a^2b+2ab^2+3b^3$,
$4a^3+6a^2b-ab^2-6b^3$.

13. $a^3-b^3-c^3$, $b^3-c^3-a^3$, $c^3-a^3-b^3$, $a^3+b^3+c^3$.

14. $2a^4-3a^3b+7a^2b^2-ab^3+b^4$, $3a^4+5a^3b-8a^2b^2+9ab^3-4b^4$,
$-4a^4-2a^3b+a^2b^2-8ab^3+3b^4$.

15. $2a^4-2b^4+3c^4$, $-10a^4+7b^4-8c^4$, $6a^4-5b^4-4c^4$, $3a^4+b^4+10c^4$.

16. x^2-5x+7, $-3x^2+4x-8$, $4x^2-7x+2$.

17. $5x^2-4x+3$, $3x^2+2x-7$, $-7x^2+3x+4$.

18. $x^2-3xy+y^2$, $-2x^2-4xy+5y^2$, $4x^2+7xy-4y^2$.

19. $3y^2+4xy-x^2$, $-2y^2-5xy+3x^2$, $y^2+xy-2x^2$.

20. y^2-5y+3, $4y^2+9y-6$, $-3y^2-2y+1$.

21. $2x^2-5x+6$, $-8x^2+3x-2$, $3x^2-7x+5$, $4x^2+10x-8$.

22. $x^2-2ax+a^2$, $-3x^2+5ax-4a^2$, $8x^2-4ax-3a^2$, $-6x^2+ax+6a^2$.

23. $ab+5ax^2-3a^2x$, $-4ab-3ax^2+8a^2x$, $7ab-9ax^2-4a^2x$.

24. x^3-4x^2+7x-3, $-10x^3+8x^2-8x+9$,
$3x^3-3x^2-5x+2$, $6x^3-x^2+6x-7$.

25. $3x^4-2x^3-x$, $-4x^4+3x^2+2x$, $2x^4+3x^3-2x^2$.

26. $2x^4-7x^3+5x^2-6x+3$, $5x^4+3x^3+4x^2+9x-1$,
$-6x^4+4x^3-9x^2-3x-2$.

27. $3x^4-5x^2+1$, $7x^3+4x^2-6$, $-x^3+x^2-4x$, $-2x^4-5x^3+3x+4$.

28. $5a^2x^3-a^3x^2+4a^4x$, $3ax^4-2a^2x^3-3a^5$,
$x^5-8ax^4+7a^2x^3$, $-9a^3x^2+a^4x+2a^5$.

29. $2y^3-3y^2+2y-6$, $-5y^3+2y^2-5y+2$, $4y^3+4y+3$.

30. $3+a-a^3-4a^2$, $4a+3a^3-a^2-5$,
$2a^3-3a^2+6-4a$, $9a^2+3-5a+5a^3$.

31. $3a^3-4a^2+7a-2$, $3a^2-4a^3+7-2a$,
$3a-4+7a^2-2a^3$, $3-4a+7a^3-2a^2$.

32. $x^2-y^2-z^2$, $3x^2-4y^2+5z^2$, $-7x^2+9y^2-6z^2$, $4x^2-3y^2+3z^2$.

33. $2x^3-6x^2y-3xy^2-y^3$, $-5x^3+8x^2y-2xy^2+7y^3$,
$8x^3-9x^2y+6xy^2-2y^3$.

34. $4x^2-5xy+2y^2+7x-3y+4$, $-8x^2+9xy-6y^2-8x+2y-9$,
$5x^2-6xy+7y^2-3x+6y-1$.

35. $x^4-2x^3y+5x^2y^2-7xy^3+4y^4$, $3x^4+5x^3y-9x^2y^2+2xy^3-9y^4$,
$-2x^4-x^3y+2x^2y^2+3xy^3+7y^4$.

36. $x^3+x^2y-x^2z+xy^2-xyz$, $x^2z-xy^2+xz^2-y^2z+y^3$,
$-x^2y-xz^2+xyz+y^2z+z^3$.

37. $\frac{1}{4}a-\frac{1}{3}b$, $\frac{1}{5}a+\frac{1}{6}b$, $\frac{1}{20}a+\frac{1}{15}b$.

38. $\frac{1}{6}a+\frac{1}{12}b-\frac{1}{12}c$, $-\frac{1}{2}a+\frac{1}{4}b+\frac{1}{4}c$, $\frac{1}{3}a-\frac{1}{3}b-\frac{1}{6}c$.

39. $\frac{1}{5}x^2-\frac{1}{4}x+\frac{1}{8}$, $\frac{1}{10}x^2-\frac{1}{3}x+\frac{1}{4}$, $-\frac{1}{2}x^2+\frac{1}{2}x-\frac{1}{2}$.

40. $\frac{3}{5}x^2-\frac{2}{3}xy-\frac{2}{7}y^2$, $-\frac{1}{3}x^2+\frac{3}{4}xy-\frac{2}{3}y^2$, $\frac{11}{15}x^2-\frac{1}{12}xy-\frac{1}{21}y^2$.

41. $\frac{1}{2}zx-\frac{5}{6}xy$, $\frac{2}{3}xy-\frac{22}{45}yz$, $\frac{3}{5}yz-\frac{5}{14}zx$.

42. $x^2-\frac{1}{2}y^2-\frac{1}{2}z^2$, $y^2-\frac{1}{2}z^2-\frac{1}{2}x^2$, $z^2-\frac{1}{2}x^2-\frac{1}{2}y^2$.

43. If $x=3a-b+6c$, $y=5b-7c+4a$, $z=2c-6a-3b$,
prove that $x+y+z=a+b+c$.

44. If $x=\frac{1}{2}p-\frac{1}{3}q+\frac{1}{5}r$, $y=\frac{1}{12}q-\frac{1}{4}r-\frac{1}{3}p$, $z=\frac{1}{20}r-\frac{1}{6}p+\frac{1}{4}q$,
shew that $x+y+z=0$.

45. If $u=3a^2-2ca+5ab-5b^2$, $v=6b^2-4ab+7bc-8c^2$,
$w=9c^2-6bc+3ca-2a^2$,
prove that $u+v+w=a^2+b^2+c^2+bc+ca+ab$.

46. A man started in business and lost Rs. 200 in his first year and Rs. 100 in his second year, while he gained Rs. 300 in his third year; what were his gains at the end of his first, second and third year, respectively?

47. A penniless man borrows £x from a friend and repays the loan at the rate of £y per month; what is the amount of his property at the end of z months?

48. A traveller walks due north eight miles, then due south ten miles, and then again due north two miles; where is he? Trace his progress algebraically.

49. A man walking at the rate of 12 miles a day walks due east for 5 days and then due west for $5\frac{1}{2}$ days; express algebraically his distance from his original starting-point.

50. A boy twenty minutes late for school arrived there at a quarter past 12 noon; at what time P.M. was he due?

CHAPTER III.

SUBTRACTION.

22. SUBTRACTION is the reverse of addition and is performed by changing the signs of all quantities which are to be subtracted and then proceeding as in addition. The change need not be expressed but may be supposed only.

This rule may be obtained by considering what takes place when the operation of subtraction is being performed. If we are required to subtract $8+5$ from 15, it is plainly immaterial whether we first subtract 8 from 15 and then 5 from what remains, or at once subtract the sum $8+5$ from 15, the result in each case being 2; hence

$$15-(8+5)=15-8-5.$$

Similarly on subtracting $b+c$ from a, the result is obviously the same whether we first subtract b from a and then c from what remains, or at once subtract the sum $b+c$; hence

$$a-(b+c)=a-b-c.$$

Again suppose we are required to subtract $8-5$ from 15. By subtracting 8 from 15 we have obviously subtracted too much by 5, since we are required to subtract not 8 but 8 diminished by 5; hence 5 must be added to the remainder, and we have

$$15-(8-5)=15-8+5;$$

the value of each result after the operations indicated being 12.

Similarly on subtracting $b-c$ from a, if we subtract b only we have obviously subtracted too much by the quantity c, since we are required to subtract not b but b diminished by c; hence c must be added to the remainder, and we have

$$a-(b-c)=a-b+c.$$

This method of proof may be extended to compound expressions containing as many terms as we please, and we may prove similarly that

$$a-(b-c-d)=a-b+c+d,$$

and so on.

In all these cases we see that to subtract an expression is the same as to add the expression with the sign of every term changed, and we are thus led to the rule given at the beginning of the chapter.

23. For example let it be required to subtract $-2a+3b-5c$ from $a-4b+6c$ and $a^3-5a^2b+8ab^2-2b^3$ from $3a^3-9a^2b-6ab^2+b^3$.

Changing the sign of every term in the expressions to be subtracted and proceeding as in addition we have

$$\begin{array}{r} a-4b+\ 6c \\ 2a-3b+\ 5c \\ \hline 3a-7b+11c \end{array} \qquad \begin{array}{r} 3a^3-9a^2b-\ 6ab^2+\ b^3 \\ -\ a^3+5a^2b-\ 8ab^2+2b^3 \\ \hline 2a^3-4a^2b-14ab^2+3b^3. \end{array}$$

The rule for fractional coefficients is the same. For instance, let it be required to subtract $\frac{1}{12}x-\frac{1}{6}y-\frac{3}{10}z$ from $\frac{1}{4}x-\frac{2}{3}y+\frac{1}{5}z$. Proceeding as before we have

$$\begin{array}{r} \frac{1}{4}x-\frac{2}{3}y+\ \frac{1}{5}z \\ -\frac{1}{12}x+\frac{1}{6}y+\frac{3}{10}z \\ \hline \frac{1}{6}x-\frac{1}{2}y+\ \frac{1}{2}z. \end{array}$$

The signs need not however be changed in the actual working of examples, but it will suffice to suppose them changed and treat them accordingly.

24. We are thus also led to the following rule regarding the use of brackets.

When an expression within brackets is preceded by the – sign, the brackets may be removed on changing the sign of every term within them.

And conversely:

Any expression may be put within brackets and preceded by the – sign, if the sign of every term to be included within them be changed.

Brackets may be removed or introduced to any extent that suits our convenience provided these rules and those of Art. 17 be observed. On removing brackets it is desirable to commence

with the innermost pair, then to proceed with the innermost pair of those which remain, and so on until all are removed. Thus

$$\begin{aligned} & a-[b-\{c-(d-\overline{e-f})\}] \\ =\ & a-[b-\{c-(d-e+f)\}] \\ =\ & a-[b-\{c-d+e-f\}] \\ =\ & a-[b-c+d-e+f] \\ =\ & a-b+c-d+e-f. \end{aligned}$$

Brackets may also frequently be introduced in various ways in the same expression. Thus

$$\begin{aligned} & a-b+c-d+e-f \\ =\ & a-\{b-c+(d-e+f)\} \\ \text{or} \quad =\ & a-b+\{c-(d-e+f)\} \\ \text{or} \quad =\ & \{a-(b-c)\}-\{d-(e-f)\}, \end{aligned}$$

and so on.

25. Subtraction in algebra is performed by prefixing the $-$ sign to the quantity to be subtracted, and the result of subtracting b, i.e. $+b$, from a is $a-b$, so that

$$-(+b)=-b.$$

Again, we have seen above (Art. 22) that the result of subtracting a negative quantity like $-b$ from a and therefore of prefixing the $-$ sign to $-b$ is $a+b$, so that

$$-(-b)=+b.$$

Hence a negative sign operating on a positive sign produces a negative sign and operating on a negative sign produces a positive sign.

These results may also be obtained independently by considering that subtraction is the reverse of addition, and that therefore to add and subtract the same quantity produces no effect; hence

$$a=a+b-b.$$

Again, if equals are taken away from equals, the remainders are equal; therefore

$$a-(+b)=a+b-b-(+b)=a-b;$$

and

$$a-(-b)=a+b-b-(-b)=a+b;$$

since in the first case $+b$ and $-(+b)$, and in the second case $-b$ and $-(-b)$, cancel each other, the same quantity being added and subtracted.

We notice that to subtract the negative quantity $-b$ is the same as to add the positive quantity b.

EXAMPLES.

Subtract

1. $a-b$ from $a+b$.
2. $2a-3b$ from $3a-2b$.
3. $a-b-c$ from $a+b+c$.
4. $3a-b+2c$ from $2a+b-3c$.
5. $4a-c$ from $a-3b+2c$.
6. $-a+b-c$ from $a-b-3c$.
7. $-\frac{1}{4}a-\frac{1}{5}b+\frac{1}{3}c$ from $\frac{3}{4}a-\frac{1}{10}b+\frac{1}{12}c$.
8. $4x-5y$ from $3x-4y$.
9. $-x+y$ from $x-y$.
10. $3x-6y-7z$ from $3x+2y-4z$.
11. $x-y$ from $5x-6y-2z$.
12. $-4x-7z$ from $-x+3y-4z$.
13. $5x-y-6z$ from $x+3y$.
14. $\frac{1}{3}x-\frac{3}{5}y+\frac{1}{2}z$ from $\frac{1}{2}x-\frac{2}{3}y+z$.
15. $-\frac{1}{2}x-\frac{1}{4}y+\frac{2}{3}z$ from $x-\frac{1}{2}y-\frac{1}{3}z$.

From

16. x^2-2x+1 take $3x^2-1$.
17. $5x^2-3$ take $4x^2-x-2$.
18. $2x^2+ax-bx-b^2$ take x^2-bx-a^2.
19. $ax^2-bx+a^2-b^2$ take $bx^2+ax-bx-b^2$.
20. $7x^2+4xy-8y^2$ take $-2x^2-xy-9y^2$.
21. a^2-5a-4 take $-a^2-7a-6$.
22. $1-3a+7a^2-2a^3-a^4$ take $-3+a+3a^2-6a^3-5a^4$.
23. x^3-2x^2+1 take $-x^3-2x+2$.
24. $2x^3-x$ take x^3-x^2+x.
25. $-x^4+a^2x^2-a^4$ take $-2x^4-ax^3+a^3x$.
26. $yz+zx+xy$ take $-yz+zx-xy$.
27. $a^2-b^2+c^2+bc-2ca$ take $c^2+a^2-b^2-bc-2ab$.
28. $a^3-3a^2x+3ax^2-x^3$ take $-3a^3+a^2x-ax^2+3x^3$.
29. $\frac{2}{3}a^2-\frac{4}{5}ab+\frac{3}{4}b^2$ take $\frac{1}{12}a^2-\frac{1}{15}ab-\frac{1}{8}b^2$.
30. $\frac{2}{5}qr-\frac{3}{4}rp+\frac{1}{6}pq$ take $-\frac{1}{3}qr+\frac{3}{2}rp-\frac{2}{9}pq$.

Simplify by removing brackets

31. $10-[6-\{1-(7-\overline{5-2})\}]$.
32. $a-[b-\{a-(b-\overline{a-b})\}]$.
33. $3a-[10a+8b-\{7b-(2a-3b)\}]-[6b-5a-\{7a-(3a-4b)\}]$.

34. $5a-[3a-\{4a-(5a-\overline{6a-7a})\}]$.

35. $a-[2b-\{3a-b-(\overline{7a-5b}-\overline{3a-2b})\}]$.

36. $a^2-[(b^2-c^2)-\{a^2-(2a^2-\overline{b^2-c^2})\}]$.

37. $1-[1-x-\{1-(1-\overline{1-x})\}]$.

38. $x^4-[2x^3-\{3x^2-(4x-1)\}]-[x^4-\{3x^3-(4x^2-5x+2)\}]$.

39. $x-[4y-\{3x-(5y-x)\}]-[3x-\{2y-(\overline{5x-10y}-\overline{4x-2y})\}]$.

40. $y^2-x^2-[z^2-\{x^2-(y^2-z^2-\overline{x^2-y^2-z^2})\}]$.

If $x=1$, $y=-1$, $z=-2$, find the value of

41. $x+y+z$. 42. $-x-y-z$. 43. $x-(-y)+(-z)$.

44. $2x+7y-3z$. 45. $-6x-10y+2z$.

46. What must be added to $b^2+c^2-a^2$ in order that the sum may be $a^2+2b^2-3c^2$?

47. What must be subtracted from

$$x^4-3x^3y+4x^2y^2-2xy^3+y^4$$

in order that the remainder may be $x^3y+x^2y^2+xy^3$?

48. To what expression must a^2-ab-b^2 be added in order that the sum may be $3a^2-4ab+b^2$?

49. Find the difference between

$$x^3-4x^2y+5xy^2-y^3 \text{ and } -x^3+3x^2y-2xy^2-3y^3.$$

50. Express algebraically: The excess of x over y is less than z by 2.

CHAPTER IV.

MULTIPLICATION.

26. JUST as the sum of any number of quantities is the same in whatever order they are placed, so the product of any number of factors is the same in whatever order they are taken. For 2×5 means that 2 is to be taken 5 times and so

$$= 2 + 2 + 2 + 2 + 2$$

$$= 10;$$

and 5×2 means that 5 is to be taken twice and

$$= 5 + 5 = 10.$$

This can also be readily proved by geometry. Let $ABCD$ be a rectangle of which one side AB contains 5 units of length and the other side AD 2 units of length, and complete the figure, so that the whole area of the rectangle is seen to be made up of ten equal squares.

Then the area $ABCD$, which is 10 square units of area, is obviously expressed by five times the area $AGHD$ or by twice the area $ABEF$, where $AGHD$ contains 2 square units and $ABEF$ contains 5 square units.

Hence $2 \times 5 = 5 \times 2$.

In the same way the product $a \times b$ means that a is to be taken b times and $b \times a$ means that b is to be taken a times; and by the same reasoning we prove that

$$a \times b = b \times a,$$

or

$$ab = ba.$$

This is true whether the factors be simple or compound, so that $(a+b)p = p(a+b)$ and $(a+b)(c+d) = (c+d)(a+b)$ and so on.

Having thus shewn that the product of any two quantities is the same in whatever order they are taken, the same follows for the product of three quantities. For by reversing the order of two factors in succession we have

$$a \times b \times c = b \times a \times c = b \times c \times a = c \times b \times a = c \times a \times b = a \times c \times b;$$

i.e.

$$abc = bac = bca = cba = cab = acb.$$

This method of proof may be extended to any number of factors.

27. When the quantities to be multiplied together are powers of the same quantity, multiplication is effected by adding the indices of their powers.

Thus $a^3 = a \times a \times a,$

and $a^5 = a \times a \times a \times a \times a.$

Therefore $a^3 \times a^5 = a \times a \times a \times a \times a \times a \times a \times a$

$$= a^8 = a^{3+5}.$$

Generally a^m means that a is to be repeated m times and

$$= a \times a \times a \times \ldots\ldots\ldots \text{ to } m \text{ factors};$$

and a^n means that a is to be repeated n times and

$$= a \times a \times a \times \ldots\ldots\ldots \text{ to } n \text{ factors.}$$

Therefore $a^m \times a^n$

$$= a \times a \times a \times \ldots\ldots\ldots \text{ to } m \text{ factors}$$

$$\times a \times a \times a \times \ldots\ldots\ldots \text{ to } n \text{ factors}$$

$$= a \times a \times a \times \ldots\ldots\ldots \text{ to } m+n \text{ factors}$$

$$= a^{m+n}.$$

28. If simple factors with numerical coefficients are to be multiplied together, the coefficients should first be multiplied together as in arithmetic, and this product should be prefixed to the product of the letters.

Thus the product of $5a^2bc$ and $3ab^3$

$$= 5 \times a^2 \times b \times c \times 3 \times a \times b^3$$
$$= 5 \times 3 \times a^2 \times a \times b \times b^3 \times c$$
$$= 15 \times a^3 \times b^4 \times c = 15a^3b^4c;$$

and the product of

$$\tfrac{2}{3}x^2 \text{ and } \tfrac{6}{7}y = \tfrac{2}{3} \times \tfrac{6}{7} \times x^2 \times y = \tfrac{4}{7} \times x^2 \times y = \tfrac{4}{7}x^2y.$$

29. The case of compound factors is a little more complicated. The value of $5(a+b)$ is five times $a+b$, i.e. $a+b$ is to be repeated 5 times; hence

$$5(a+b) = a+b+a+b+a+b+a+b+a+b$$
$$= 5a + 5b.$$

By the same reasoning $p(a+b)$ is p times $a+b$, i.e. $a+b$ is to be repeated p times, so that

$$p(a+b) = a+b+a+b+a+b+ \ldots\ldots \text{ taken } p \text{ times}$$
$$= a+a+a+ \ldots\ldots\ldots\ldots \text{ taken } p \text{ times}$$
$$+b+b+b+ \ldots\ldots\ldots\ldots \text{ taken } p \text{ times}$$
$$= pa + pb.$$

Similarly also $p(a-b)$ means that $a-b$ is to be repeated p times and

$$= a-b+a-b+a-b+ \ldots\ldots \text{ taken } p \text{ times}$$
$$= a+a+a+ \ldots\ldots\ldots\ldots \text{ taken } p \text{ times}$$
$$-(b+b+b+ \ldots\ldots\ldots\ldots \text{ taken } p \text{ times})$$
$$= pa - pb,$$

and so on; and the method of proof may be extended to a compound expression containing any number of terms.

30. When two compound factors, as $a+b$ and $c+d$, are to be multiplied together, let $c+d=p$; then as proved above

$$(a+b)(c+d) = p(a+b)$$
$$= pa + pb = (c+d)a + (c+d)b$$
$$= ac + ad + bc + bd.$$

In the same manner

$$(a-b)(c+d) = p(a-b) = pa - pb$$
$$= (c+d)a - (c+d)b = ac + ad - bc - bd.$$

Let $c-d=q$; then

$$(a-b)(c-d)=q(a-b)=qa-qb$$
$$=(c-d)a-(c-d)b=ac-ad-(bc-bd)$$
$$=ac-ad-bc+bd.$$

In these results we notice that the term ac, i.e. $+ac$, in the product corresponds to the terms a and c, i.e. $+a$ and $+c$ in the factors, and similarly that $+ad$ corresponds to $+a$ and $+d$, $-ad$ to $+a$ and $-d$, $-bc$ to $-b$ and $+c$, and $+bd$ to $-b$ and $-d$ as well as to $+b$ and $+d$. We are hence led to a general law in multiplication which is called the **Law of Signs** and may be enunciated as follows :

Like signs multiplied together produce the + sign, unlike signs the − sign.

In accordance with this law we see that the product of x and y or of $-x$ and $-y$ is xy, and of x and $-y$ or of $-x$ and y is $-xy$; and similarly that the product of $3a$ and $2b$ or of $-3a$ and $-2b$ is $6ab$, and of $3a$ and $-2b$ or of $-3a$ and $2b$ is $-6ab$. Also by the same law we observe that every even power of -1 is 1 and every odd power is -1 ; for $(-1)^2=-1\times-1=1$, $(-1)^3=(-1)^2\times-1=1\times-1=-1$, $(-1)^4=(-1)^3\times-1=-1\times-1=1$, and so on. And generally every even power of a negative quantity is positive and every odd power is negative; for $(-a)^2=-a\times-a=a^2$, $(-a)^3=(-a)^2\times-a=a^2\times-a=-a^3$, $(-a)^4=(-a)^3\times-a=-a^3\times-a=a^4$, and so on. Or it may be noticed that $-a=-1\times a$ so that $(-a)^n=(-1)^n\times a^n$; and $(-1)^n$ is 1 or -1 according as n is even or odd, from which the result follows at once.

It will be observed that the Law of Signs embraces the results obtained in Arts. 20, 25 ; for

$$+(+b),\ +(-b),\ -(+b),\ -(-b)$$

are equivalent to

$$+1\times+b,\ +1\times-b,\ -1\times+b,\ -1\times-b.$$

31. We are now in a position to state a general Rule for multiplying together two algebraical expressions :

Multiply each term of one expression by each term of the other ; if the terms have like signs prefix to the product the + sign, if unlike the − sign; these products when collected form the complete product.

The process of multiplication will be much simplified by an orderly disposition of terms in the two expressions, and by arranging them so that like terms may stand in the same column. We shall now obtain a few results which occur so frequently in multiplication that they should be treated as types and stored in the memory for future use.

The square of $a+b$ is $a^2+2ab+b^2$ and of $a-b$ is $a^2-2ab+b^2$, and the product of $a+b$ and $a-b$ is a^2-b^2.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ \quad ab+b^2 \\ \hline a^2+2ab+b^2 \end{array} \qquad \begin{array}{r} a-b \\ a-b \\ \hline a^2-ab \\ \quad -ab+b^2 \\ \hline a^2-2ab+b^2 \end{array} \qquad \begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2 \qquad -b^2. \end{array}$$

The product of $a+b$ and a^2-ab+b^2 is a^3+b^3 and of $a-b$ and a^2+ab+b^2 is a^3-b^3.

$$\begin{array}{r} a^2-ab+b^2 \\ a+b \\ \hline a^3-a^2b+ab^2 \\ \quad a^2b-ab^2+b^3 \\ \hline a^3 \qquad +b^3 \end{array} \qquad \begin{array}{r} a^2+ab+b^2 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ \quad -a^2b-ab^2-b^3 \\ \hline a^3 \qquad -b^3 \end{array}$$

The five results just obtained are here collected together for purposes of reference and should be committed to memory.

$$(a+b)^2 = a^2+2ab+b^2 \quad \ldots\ldots\ldots(1),$$

$$(a-b)^2 = a^2-2ab+b^2 \quad \ldots\ldots\ldots(2),$$

$$(a+b)(a-b) = a^2-b^2 \quad \ldots\ldots\ldots\ldots\ldots(3),$$

$$(a+b)(a^2-ab+b^2) = a^3+b^3 \quad \ldots\ldots\ldots\ldots\ldots(4),$$

$$(a-b)(a^2+ab+b^2) = a^3-b^3 \quad \ldots\ldots\ldots\ldots\ldots(5).$$

Other results which do not so often occur but which it is still useful to remember are that the cube of $a+b$ is

$$a^3+3a^2b+3ab^2+b^3$$

and of $a-b$ is $a^3-3a^2b+3ab^3-b^3$.

The square of $a+b$ has just been proved to be $a^2+2ab+b^2$ and of $a-b$ to be $a^2-2ab+b^2$; then to find the cube of $a+b$ we must multiply $a^2+2ab+b^2$ by $a+b$, and to find the cube of $a-b$ we must multiply $a^2-2ab+b^2$ by $a-b$. Thus

$$\begin{array}{l} a^2+2ab+b^2 \\ a+b \\ \hline a^3+2a^2b+ab^2 \\ \qquad a^2b+2ab^2+b^3 \\ \hline a^3+3a^2b+3ab^2+b^3 \end{array} \qquad \begin{array}{l} a^2-2ab+b^2 \\ a-b \\ \hline a^3-2a^2b+ab^2 \\ \qquad -a^2b+2ab^2-b^3 \\ \hline a^3-3a^2b+3ab^2-b^3 \end{array}$$

32. Multiplication will be greatly facilitated by using one or other of these formulæ, the formula to be used being suggested by the shape of the expressions under treatment, and it will be found that in a large number of cases actual multiplication can be avoided and the result written down at once by the aid of a formula. Thus let it be required to find the product of x^2+xy+y^2 and x^2-xy+y^2.

$$\begin{aligned}(x^2+xy+y^2)\,(x^2-xy+y^2) &= \{(x^2+y^2)+xy\}\,\{(x^2+y^2)-xy\} \\ &= (x^2+y^2)^2-x^2y^2 \text{ by (3)} \\ &= x^4+2x^2y^2+y^4-x^2y^2 \text{ by (1)} \\ &= x^4+x^2y^2+y^4.\end{aligned}$$

This result is frequently useful.

In the same way

$$\begin{aligned}(x^3-y^3)\,(x^2-xy+y^2)\,(x+y) &= (x^3-y^3)\,(x^3+y^3) \text{ by (4)} \\ &= x^6-y^6 \text{ by (3)}.\end{aligned}$$

$$\begin{aligned}(x+y+z)^2 &= \{x+(y+z)\}^2 = x^2+2x\,(y+z)+(y+z)^2 \text{ by (1)} \\ &= x^2+2xy+2zx+y^2+2yz+z^2 \text{ by (1)} \\ &= x^2+y^2+z^2+2\,(yz+zx+xy).\end{aligned}$$

$$\begin{aligned}(x+y+z)^3 &= \{x+(y+z)\}^3 \\ &= x^3+3x^2\,(y+z)+3x\,(y+z)^2+(y+z)^3 \\ &= x^3+3x^2\,(y+z)+3x\,(y^2+2yz+z^2)+y^3+3y^2z+3yz^2+z^3 \\ &= x^3+y^3+z^3+3x^2\,(y+z)+3y^2\,(z+x)+3z^2\,(x+y)+6xyz.\end{aligned}$$

It will be observed how much the labour of the student is lightened by using formulæ instead of going through the drudgery of actual multiplication, and it is important that he should as early as possible acquire a readiness in the application of such formulæ.

33. When two expressions to be multiplied together involve a common letter, their terms should be so arranged as to be in ascending or descending powers of that letter. Thus it is much

better to multiply $x-a$ by $x-b$ under that form than under the form $-b+x$, and to multiply x^2-ax+b by x^2-cx+d under that form than under either of the forms $d+x^2-cx$ or $-cx+d+x^2$, the labour incurred being less if the terms are arranged as suggested.

With this arrangement we have

$$\begin{array}{l} x-a \\ x-b \\ \hline x^2-ax \\ \quad -bx+ab \\ \hline x^2-(a+b)\,x+ab \end{array}$$

and we notice that in the product of two binomials of the form $x-a$ and $x-b$ the coefficient of x is the sum of the second terms of the binomials, and that the term independent of x is their product. Again, if we multiply this product by another binomial factor $x-c$ we obtain the product of $x-a$, $x-b$ and $x-c$.

$$\begin{array}{llll} x^2-(a+b)\,x & +ab & & \\ x-c & & & \\ \hline x^3-(a+b)\,x^2 & +abx & & \\ \quad -cx^2 & +(ca+bc) & x-abc & \\ \hline x^3-(a+b+c)\,x^2+(bc+ca+ab)\,x-abc. & & & \end{array}$$

In this product we notice that the coefficient of x^2 is the sum of the second terms of the binomials, that the coefficient of x is the sum of the second terms multiplied two and two together, and that the term independent of x is the product of the second terms. These results are worth noticing.

34. There is a numerous class of examples in algebra in which we are required to prove that expressions are equal to each other under certain given conditions. In all such problems, after the multiplication is performed, the arrangement of terms according to ascending or descending powers of a common letter should be scrupulously observed.

Example 1. If $2s=a+b+c$, prove that

$$2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)=abc.$$

We have

$$\begin{array}{lllll} 2(s-a)(s-b)(s-c)= & 2s^3-2(a+b+c)\,s^2+2(bc+ca+ab)\,s-2abc & & & \\ a(s-b)(s-c)= & & as^2-a(b+c)\,s & + & abc \\ b(s-c)(s-a)= & & bs^2-b(c+a)\,s & + & abc \\ c(s-a)(s-b)= & & cs^2-c(a+b)\,s & + & abc \\ \hline & 2s^3-(a+b+c)\,s^2 & & + & abc. \end{array}$$

Adding up the expressions on the right we get

$2s^3-(a+b+c)\,s^2+abc$. i.e. $2s^3-2s\times s^2+abc$, i.e. $2s^3-2s^3+abc$, i.e. abc.

Example 2. If $s=a+b+c$, prove that

$$(s-3a)^2+(s-3b)^2+(s-3c)^2=3\{(b-c)^2+(c-a)^2+(a-b)^2\}.$$

We have on squaring out and arranging the terms in descending powers of s,

$$\begin{aligned}
&(s-3a)^2+(s-3b)^2+(s-3c)^2\\
&=3s^2-6(a+b+c)s+9(a^2+b^2+c^2)\\
&=3s^2-6s^2+9(a^2+b^2+c^2)\\
&=9(a^2+b^2+c^2)-3s^2\\
&=9(a^2+b^2+c^2)-3(a+b+c)^2\\
&=9(a^2+b^2+c^2)-3(a^2+b^2+c^2+2bc+2ca+2ab)\\
&=3(b^2-2bc+c^2+c^2-2ca+a^2+a^2-2ab+b^2)\\
&=3\{(b-c)^2+(c-a)^2+(a-b)^2\}.
\end{aligned}$$

EXAMPLES.

Multiply

1. $2a$ by $7a$.
2. $4a^2$ by $-5a^2$.
3. $\frac{2}{3}a$ by $-\frac{9}{8}a^3$.
4. $-8x$ by $4x^3$.
5. $-6x$ by $-3x^4$.
6. $\frac{2}{5}x$ by $-\frac{1}{2}x^2$.
7. $x-2$ by $x-5$.
8. $x-1$ by $x+6$.
9. $x-4$ by $x+10$.
10. x^2-3x+2 by $x+2$.
11. $x^2-4x+16$ by $x+4$.
12. x^2+3x-4 by $x-3$.
13. $x-2y$ by $3y$.
14. $\frac{1}{3}x-\frac{1}{4}y$ by $\frac{3}{4}y$.
15. $\frac{5}{6}x-\frac{2}{3}y$ by $6x-\frac{3}{2}y$.
16. $2a-3b$ by $2b-3a$.
17. $4a^2-7b^2$ by $2a-5b$.
18. $5a^2-ab$ by a^2-2ab.
19. $3x-4y$ by $5y-2x$.
20. $\frac{2}{5}x-\frac{3}{4}y$ by $\frac{10}{3}x-\frac{5}{6}y$.
21. $5x^2-3y^2$ by $2x-y$.
22. $a^2-4ab-5b^2$ by $7b-3a$.
23. $3b^2-2ab-a^2$ by $4a-3b$.
24. $2x^2-3xy-y^2$ by $2x-4y$.
25. $-3x^2+4xy-2y^2$ by $3y-x$.
26. a^3+a^2+a+1 by $a-1$.
27. $4a^3-3a^2+5a-2$ by $2a-3$.
28. $\frac{1}{2}a^3-\frac{1}{3}a^2+\frac{1}{4}a-\frac{1}{6}$ by $12a-12$.
29. $x^3-4x^2+5x-24$ by x^2-4x-5.
30. $2x^2-3y^2+4xy-x+2y$ by $x-2y$.
31. $x^2-y^2-xy+x-y-1$ by $x-y+1$.
32. $a^2x^2-2a^3x-5a^4$ by $2x-a$.

33. $4a^2-3bx+5abx^2$ by $a^2-bx-b^2x^2$.

34. $x^3-ax^2+a^2x-a^3$ by $x+a$.

35. $x^2+y^2+z^2-yz-zx-xy$ by $x+y+z$.

36. $a^2-3ab+2b^2$ by $2a^2+ab-3b^2$.

37. $x^2-xy+2y^2$ by $y^2+xy-2x^2$.

38. $x^2-(a-b)x+b^2$ by $x^2+(a+b)x-a^2$.

39. $x^3-2ax^2+3a^2x+a^3$ by x^2+ax-a^2.

40. $x^4+x^3y+x^2y^2+xy^3+y^4$ by x^2-y^2.

By means of formulæ find the product of the following:

41. $(7x-8y)(7x+8y)$.

42. $(x+y+z)(x-y+z)$.

43. $(4x-3y+2z)(4x+3y-2z)$.

44. $(x^2+x+1)(x-1)$.

45. $(x^2-x+1)(x+1)$.

46. $(4a^2-6ab+9b^2)(2a+3b)$.

47. $(25a^2+10ab+4b^2)(5a-2b)$.

48. $(a^2-b^2+a+b)(a^2+b^2+a-b)$.

49. $(a^2-b^2-2bc-c^2)(a^2+b^2+2bc-c^2)$.

50. $(2ab-a^2+b^2-c^2)(2ab+a^2-b^2+c^2)$.

51. $(a^3-b^3+a^2b+ab^2)(a^3+b^3+a^2b-ab^2)$.

52. $(2x^2+2x+1)(2x^2-2x+1)$.

53. $(x^2+3xy-2y^2)(x^2-3xy+2y^2)$.

54. $(x^2+\sqrt{5}xy+y^2)(x^2-\sqrt{5}xy+y^2)$.

55. $(\sqrt{3}xy-x^2-y^2)(\sqrt{3}xy+x^2+y^2)$.

56. $(49x^2-42xy+9y^2)(49x^2+42xy+9y^2)$.

57. $(x^3-x^2+x-1)(x^3+x^2+x+1)$.

58. $(a^3+4a^2b+2ab^2+b^3)(a^3-4a^2b+2ab^2-b^3)$.

59. $(a^2-2ab-ab-2b^2)(a+2b)$.

60. $(a^2+ab+b^2)(a^2-ab+b^2)(a^4-a^2b^2+b^4)$.

61. $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

62. $(a^2-b^2)(a^2+b^2)(a^4+b^4)$.

63. $(x^2-x+1)(x^2+x+1)(x^4-x^2+1)$.

64. $(x^{2n}-2x^n+1)(x^{2n}+2x^n+1)(x^{4n}+2.x^{2n}+1)$.

65. $(x^2-xy+y^2)(x^2+xy+y^2)(x^2-y^2)$.

Find by formulæ:

66. $(3a-4b)^2$.

67. $(5x^2+7y^3)^2$.

68. $(a-b+c)^2$.

69. $(3a+2b)^3$.

70. $(x-5y)^3$.

If $s=a+b+c$, prove that

71. $(s-2b)(s-2c)+(s-2c)(s-2a)+(s-2a)(s-2b)$
$=2(bc+ca+ab)-(a^2+b^2+c^2)$.

72. $(as+bc)(bs+ca)(cs+ab)=(b+c)^2(c+a)^2(a+b)^2$.

73. $(s-a)^2+(s-b)^2+(s-c)^2-s^2=a^2+b^2+c^2$.

If $2s=a+b+c$, prove that

74. $s^2+(s-a)^2+(s-b)^2+(s-c)^2=a^2+b^2+c^2$.

75. $s^2-(s-b)(s-c)-(s-c)(s-a)-(s-a)(s-b)=\frac{1}{2}(a^2+b^2+c^2)$.

76. $s^3-(s-a)^3-(s-b)^3-(s-c)^3=3abc$.

If $3s=a+b+c$, prove that

77. $3(s-b)(s-c)+3(s-c)(s-a)+3(s-a)(s-b)$
$=bc+ca+ab-a^2-b^2-c^2$.

78. $(s-a)^3+(s-b)^3+(s-c)^3=3(s-a)(s-b)(s-c)$.

79. $(s-a)^4+(s-b)^4+(s-c)^4=2(s-b)^2(s-c)^2$
$+2(s-c)^2(s-a)^2+2(s-a)^2(s-b)^2$.

If $a+b+c=0$, prove that

80. $a(a+b)(a+c)=b(b+c)(b+a)=c(c+a)(c+b)$.

81. $a^3+b^3+c^3=3abc$.

82. $4(b^2c^2+c^2a^2+a^2b^2)=(a^2+b^2+c^2)^2$.

83. By means of the last example shew that
$(x-y)^2(z-x)^2+(y-z)^2(x-y)^2+(z-x)^2(y-z)^2$
$=(x^2+y^2+z^2-yz-zx-xy)^2$.

84. If $a+b+c+d=0$, prove that
$(a+b)(a+c)(a+d)=(b+c)(b+d)(b+a)$
$=(c+d)(c+b)(c+a)=(d+a)(d+c)(d+b)$.

85. Shew that the expression
$$x^2+y^2+z^2-yz-zx-xy$$
is not changed in value by increasing or diminishing each of the quantities x, y and z by any the same quantity.

Prove the truth of the following results:

86. $a(b-c)+b(c-a)+c(a-b)=0$.

87. $(b-c)(a-b-c)+(c-a)(b-c-a)+(a-b)(c-a-b)=0$.

88. $(-a+b+c+d)(a-b+c+d)$
$+(a+b-c+d)(a+b+c-d)=4(ab+cd)$.

89. $(x+y)^3+(x-y)^3+3(x+y)^2(x-y)+3(x-y)^2(x+y)=8x^3$.

90. $(x+y+z)^3-(y+z-x)^3-(z+x-y)^3-(x+y-z)^3=24xyz.$

91. If $(x+1)^2=x$, evaluate $11x^3+8x^2+8x-2$ and shew that $x^3=1$.

92. If $(x-1)^2=-x$, evaluate $9x^3-4x^2+4x+5$ and shew that $x^3=-1$.

93. If $(x-2)^2=3$, evaluate $8x^3-23x^2-28x+9$.

94. If $(x+1)^3=3x^2+2x$, evaluate $5x^4-7x^3+5x^2-2x-6$.

95. If $(x+1)^3=4x^2+3x$, evaluate $2x^4+x^3-3x^2+2x+4$.

If $x=-\frac{1}{3}$, $y=-2$, find the value of

96. $\dfrac{9x^3+6x^2-2x}{6x^2+2x+1}$.

97. $\dfrac{\sqrt{17-4y}-y^2}{y^2-\sqrt{1-y^3}}$.

98. $\dfrac{x+y+2xy}{x-y-4xy}$.

99. $\dfrac{3xy-6x-5y}{9x^2-15xy+4y^2}$.

100. $\dfrac{3x^3+x^2y-xy^2-y^3}{3x^3-5x^2y-12xy^2+y^3}$.

CHAPTER V.

DIVISION.

35. Division is the inverse operation of multiplication, for whereas in multiplication we are to find the product of two given factors, in division the product and one factor are given and the other factor is to be determined. The given product is called the **dividend**, the given factor the **divisor**, and the factor which is to be found the **quotient**. As the multiplication of factors can be performed in any order (Art. 26), so division, which is the inverse of multiplication, can also be performed in any order. Also every example in multiplication furnishes two examples at least in division.

We shall first shew that the Law of Signs holds for division. Let a be the product of two quantities b and c, so that we have the following results in multiplication:

$$a = bc = -b \times -c\,; \quad -a = -b \times c = b \times -c.$$

In division these statements take the form

$$c = \frac{a}{b} = \frac{-a}{-b}\,; \quad -c = \frac{a}{-b} = \frac{-a}{b}\,;$$

which proves that in division, as in multiplication, like signs produce the + sign and unlike signs the − sign.

36. To divide a power of a quantity by another power of the same quantity we must subtract the index of the power in the divisor from that in the dividend.

Thus to divide a^5 by a^2, we have

$$\frac{a^5}{a^2} = \frac{a \times a \times a \times a \times a}{a \times a} = a \times a \times a = a^3 = a^{5-2}\,;$$

or thus: $\quad a^5 = a^3 \times a^2\,;$ therefore $\dfrac{a^5}{a^2} = a^3 = a^{5-2}.$

Any other example may be treated similarly, and to prove the general proposition we have, where m and n are positive integers and m is greater than n,

$$\frac{a^m}{a^n} = \frac{a \times a \times a \times \ldots \text{to } m \text{ factors}}{a \times a \times a \times \ldots \text{to } n \text{ factors}} = a \times a \times a \times \ldots \text{to } m-n \text{ factors} = a^{m-n}.$$

This general result may also be obtained thus:

$$a^m = a^{m-n+n} = a^{m-n} \times a^n \text{ ; therefore } \frac{a^m}{a^n} = a^{m-n}.$$

If the index in the dividend is equal to the index in the divisor we have

$$a^0 = a^{m-m} = \frac{a^m}{a^m} = 1 ;$$

hence *any quantity raised to the power zero is equal to unity.*

37. Having now shewn how to divide a power of a quantity by another power of that quantity and that the Law of Signs holds in division, we are in a position to divide one simple expression by another and are led to the following rule:

Divide the coefficient of the dividend by that of the divisor as in arithmetic, and the power of each letter in the dividend by that of the same letter in the divisor, and prefix the + or − sign to the quotient according as the divisor and dividend have the same sign or opposite signs.

Thus $35a^4b^5 \div 7a^3b$ or $-35a^4b^5 \div -7a^3b$

$= 5a^{4-3}b^{5-1} = 5ab^4$;

$48x^7y^4z^3 \div -16x^3y^4z$ or $-48x^7y^4z^3 \div 16x^3y^4z$

$= -3x^{7-3}y^{4-4}z^{3-1} = -3x^4z^2$;

$3x^4y^3 \div 15xy = \frac{3}{15}x^{4-1}y^{3-1}$

$= \frac{1}{5}x^3y^2$.

38. Again we have shewn (Art. 29) that

$$a(b+c) = ab + ac,$$

so that $$\frac{ab+ac}{a} = b + c ;$$

and similarly $$\frac{ab-ac}{a} = b - c,$$

and $$\frac{ab-ac+ad}{a} = b - c + d,$$

and so on.

We are hence led to the following rule for dividing a compound expression by a simple expression:

Divide each term of the dividend separately by the divisor and collect the results.

Thus $(20a^4b^3 - 8a^3b^4 - 24a^2b^5) \div 4a^2b^2$

$= 5a^{4-2}b^{3-2} - 2a^{3-2}b^{4-2} - 6a^{2-2}b^{5-2}$

$= 5a^2b - 2ab^2 - 6b^3$;

$(12x^4y^3z^2 - 3x^3y^3z^3 - 6xy^2z^6) \div -3xyz^2$

$= -4x^{4-1}y^{3-1}z^{2-2} + x^{3-1}y^{3-1}z^{3-2} + 2x^{1-1}y^{2-1}z^{6-2}$

$= -4x^3y^2 + x^2y^2z + 2yz^4.$

39. To divide a compound expression by another, we begin by arranging the dividend and divisor in descending powers of some common letter. Thus let it be required to divide a compound expression N by another compound expression a, and first let the quotient consist of two terms b, c arranged in descending powers of the common letter; then

$$N = a(b + c) = ab + ac.$$

Hence to obtain b, the first part of the quotient which results from dividing N by a, we must divide by a the first part ab of the dividend which is of the highest degree in the common letter. Having thus found b, we have

$$N - ab = ac;$$

hence to obtain c we must treat $N - ab$ as we just now treated N to obtain b, i.e. we must divide by a the part ac of the dividend which remains after subtracting ab from N.

Similarly if the quotient consist of three terms b, c, d in descending powers of the common letter, so that

$$N = a(b + c + d) = ab + ac + ad,$$

we first find b and c as just explained, and then we have the equation

$$N - ab - ac = ad.$$

Hence to obtain d we must divide by a the part ad which remains after subtracting ac from $N - ab$, i.e. which remains after subtracting $ab + ac$ from N; and so on for any number of terms in the quotient.

To illustrate this process by an example let us divide $2x^2 + xy - 3y^2$ by $x - y$. Arranging the dividend in the form

$$2x^2 - 2xy + 3xy - 3y^2,$$

with the notation just employed N stands for the whole expression and a stands for $x-y$, while ab stands for $2x^2-2xy$, i.e. for $2x(x-y)$, and ac for $3xy-3y^2$, i.e. for $3y(x-y)$. Then to get the first term b of the quotient we divide $2x^2-2xy$ by $x-y$ and obtain $2x$; and to get the second term c we subtract $2x^2-2xy$ from the whole expression $2x^2+xy-3y^2$ and divide what is left, viz. $3xy-3y^2$, by $x-y$, and we thus obtain $3y$.

The work may be arranged as follows by a method resembling that of Long Division in arithmetic:

$$\begin{array}{l} x-y\,)\,2x^2+\ xy-3y^2\,(\,2x+3y \\ \quad\ 2x^2-2xy \\ \quad\ \overline{\qquad\qquad} \\ \qquad\quad 3xy-3y^2 \\ \qquad\quad 3xy-3y^2 \\ \qquad\quad \overline{\qquad\qquad} \end{array}$$

Here by dividing $2x^2$ by x we get $2x$ the first term of the quotient. Multiplying the whole divisor by $2x$ we get $2x^2-2xy$, and subtracting this product we get $3xy-3y^2$. Dividing $3xy$ by x we get $3y$ the next term in the quotient, and multiplying the whole divisor by $3y$ we get $3xy-3y^2$, which on subtraction leaves no remainder. The operations here described are exactly equivalent to our treatment of N in the theoretical case considered, and amount to writing the dividend in the convenient form $2x^2-2xy+3xy-3y^2$, i.e. $2x(x-y)+3y(x-y)$, which on division by $x-y$ gives by inspection $2x+3y$.

40. We are thus led to the following general rule for effecting the division of one compound expression by another:

Arrange dividend and divisor in descending powers of some common letter. Divide the term on the left containing the highest power of that letter in the dividend by the term on the left containing the highest power of that letter in the divisor. Multiply the whole divisor by this quotient, put the product under the dividend, subtract and bring down as many terms as are required. Repeat this process until all the terms of the dividend have been brought down.

Example 1. Divide $18x^3-12x^2-13x+5$ by $3x-1$.

$$\begin{array}{l} 3x-1\,)\,18x^3-12x^2-13x+5\,(\,6x^2-2x-5 \\ \quad\ 18x^3-\ 6x^2 \\ \quad\ \overline{\qquad\qquad} \\ \qquad\quad -\ 6x^2-13x \\ \qquad\quad -\ 6x^2+\ 2x \\ \qquad\quad \overline{\qquad\qquad} \\ \qquad\qquad\quad -15x+5 \\ \qquad\qquad\quad -15x+5 \\ \qquad\qquad\quad \overline{\qquad\qquad} \end{array}$$

Here the process of division amounts to writing the dividend in the form

$$18x^3-6x^2-6x^2+2x-15x+5,$$

i.e. $$6x^2(3x-1)-2x(3x-1)-5(3x-1);$$

and the successive parts into which the dividend is separated and which we have denoted by ab, ac, ad, in the general proposition of last Article are $18x^3-6x^2$, $-6x^2+2x$, $-15x+5$. In accordance with that proposition we successively subtract $18x^3-6x^2$, $-6x^2+2x$, $-15x+5$ before proceeding to a fresh division, and the operation of the rule without any effort on our part splits the dividend into suitable component parts arranged in descending powers of the common letter and each containing the divisor exactly.

Example 2. Divide $3a^4-2a^3b-2a^2b^2+71ab^3-40b^4$ by $a^2-3ab+8b^2$.

$$
\begin{array}{l}
a^2-3ab+8b^2)\,3a^4-2a^3b-\ 2a^2b^2+71ab^3-40b^4\,(3a^2+7ab-5b^2 \\
\qquad\qquad 3a^4-9a^3b+24a^2b^2 \\
\qquad\qquad\qquad\qquad 7a^3b-26a^2b^2+71ab^3 \\
\qquad\qquad\qquad\qquad 7a^3b-21a^2b^2+56ab^3 \\
\qquad\qquad\qquad\qquad\qquad\quad -\ 5a^2b^2+15ab^3-40b^4 \\
\qquad\qquad\qquad\qquad\qquad\quad -\ 5a^2b^2+15ab^3-40b^4
\end{array}
$$

Here the process amounts to writing the dividend in the form

$$(3a^4-9a^3b+24a^2b^2)+(7a^3b-21a^2b^2+56ab^3)-(5a^2b^2-15ab^3+40b^4)$$

and to splitting it into three component parts each of which is exactly divisible by the divisor. And in accordance with the general proposition we successively subtract each of these three parts before proceeding to a fresh division.

41. When dealing with expressions involving several letters as a, b, c we first arrange the terms in descending powers of the first letter in alphabetical order which occurs, in this case a, then we give precedence to b over c and to a higher power of b over a lower, so that of terms involving a^3b and a^3c we should place the term in a^3b first and of terms involving a^2b^2 and a^2bc we should place the term in a^2b^2 first and so on.

Example. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

$$
\begin{array}{l}
a+b+c)\,a^3 \qquad\qquad -3abc+b^3+c^3\,(a^2-ab-ac+b^2-bc+c^2 \\
\qquad a^3+a^2b+a^2c \\
\qquad\quad -a^2b-a^2c-3abc \\
\qquad\quad -a^2b-ab^2-abc \\
\qquad\qquad\quad -a^2c+ab^2-2abc \\
\qquad\qquad\quad -a^2c\qquad\ -abc-ac^2 \\
\qquad\qquad\qquad\quad ab^2-abc+ac^2+b^3 \\
\qquad\qquad\qquad\quad ab^2\qquad\qquad\ +b^3+b^2c \\
\qquad\qquad\qquad\qquad\ -abc+ac^2\qquad -b^2c \\
\qquad\qquad\qquad\qquad\ -abc\qquad\qquad -b^2c-bc^2 \\
\qquad\qquad\qquad\qquad\qquad\quad ac^2\qquad\qquad +bc^2+c^3 \\
\qquad\qquad\qquad\qquad\qquad\quad ac^2\qquad\qquad +bc^2+c^3
\end{array}
$$

The quotient may now be written in the more symmetrical form

$$a^2+b^2+c^2-bc-ca-ab.$$

42. Sometimes the division cannot be exactly performed and in this case, as in arithmetic, there is a remainder.

Example. Divide $2a^2-15b^2$ by $a+3b$.

$$\begin{array}{l} a+3b)\ 2a^2-15b^2\ (2a-6b \\ \quad\ 2a^2+\ 6ab \\ \quad\ \overline{\quad -\ 6ab-15b^2} \\ \quad\ -\ 6ab-18b^2 \\ \quad\ \overline{\qquad\qquad 3b^2} \end{array}$$

Here $2a-6b$ is the quotient and there is a remainder $3b^2$, and the result might be expressed in either of the two forms

$$\frac{2a^2-15b^2}{a+3b}=2a-6b+\frac{3b^2}{a+3b};$$

or $$2a^2-15b^2=(a+3b)(2a-6b)+3b^2.$$

43. Problems in division may often be advantageously worked with the help of brackets.

Example. Divide

$$x^3-(2a-b+3c)x^2-(3bc-6ca-2ab)x-6abc \text{ by } x-3c.$$

$$\begin{array}{l} x-3c\big)x^3-(2a-b+3c)x^2-(3bc-6ca-2ab)x-6abc\big(x^2-(2a-b)x+2ab \\ \quad x^3\qquad\quad -3cx^2 \\ \quad\ \overline{\quad -(2a-b)x^2-(3bc-6ca-2ab)x} \\ \quad\ -(2a-b)x^2+\ 3c(2a-b)x \\ \quad\ \overline{\qquad\qquad\qquad\qquad 2abx-6abc} \\ \quad\ \qquad\qquad\qquad\qquad 2abx-6abc \end{array}$$

44. The process of division is the same whether the coefficients be integral or fractional.

Example. Divide $\frac{1}{4}x^2-\frac{43}{72}xy+\frac{1}{3}y^2$ by $\frac{3}{4}x-\frac{2}{3}y$.

$$\begin{array}{l} \frac{3}{4}x-\frac{2}{3}y)\ \frac{1}{4}x^2-\frac{43}{72}xy+\frac{1}{3}y^2\ (\frac{1}{3}x-\frac{1}{2}y \\ \quad\ \frac{1}{4}x^2-\ \frac{2}{9}xy \\ \quad\ \overline{\quad -\ \frac{3}{8}xy+\frac{1}{3}y^2} \\ \quad\ -\ \frac{3}{8}xy+\frac{1}{3}y^2 \end{array}$$

45. We have proved in the last chapter that

$$x^2-y^2=(x-y)(x+y),$$

$$x^3-y^3=(x-y)(x^2+xy+y^2);$$

similarly $$x^4-y^4=(x^2-y^2)(x^2+y^2)$$

$$=(x-y)(x+y)(x^2+y^2);$$

so that $x-y$, x^2-y^2, x^3-y^3 and x^4-y^4 are all divisible by $x-y$, while x^2-y^2 and x^4-y^4 are also divisible by $x+y$.

Also we have proved that

$$x^3+y^3=(x+y)\,(x^2-xy+y^2),$$

so that $x+y$, x^3+y^3 are divisible by $x+y$ while x^2+y^2 is found to be not so divisible, and none of these expressions are found to be divisible by $x-y$.

We can carry on these operations as far as we please and by division we find that x^5-y^5, x^6-y^6..... are all divisible by $x-y$ whether the index be even or odd, and that x^6-y^6, x^8-y^8..... are divisible by $x+y$, the index being even. We also find by trial that x^5+y^5, x^7+y^7..... are divisible by $x+y$, the index being odd, and that x^4+y^4, x^6+y^6..... are not divisible by $x+y$, the index being even; also that x^4+y^4, x^5+y^5..... are not divisible by $x-y$.

Generalizing these results we are enabled to make the following statements, which will be found to be universally true when the index is a positive integer:

x^n-y^n is always divisible by $x-y$, is divisible by $x+y$ when n is even and is not divisible by $x+y$ when n is odd;

x^n+y^n is divisible by $x+y$ when n is odd, is not divisible by $x+y$ when n is even, and is never divisible by $x-y$.

46. Examples in division can often be worked by formulæ and without the labour of actual division; in all such cases the method by formulæ is to be preferred, those most generally serviceable for the purpose being given in Art. 31.

Example 1. Divide $16x^2-40xy+25y^2$ by $4x-5y$.

$$(16x^2-40xy+25y^2)\div(4x-5y)$$
$$=(4x-5y)^2\div(4x-5y)=4x-5y.$$

Here putting $a=4x$, $b=5y$ in the formula $(a-b)^2=a^2-2ab+b^2$, the result at once follows.

Example 2. Divide $\frac{4}{9}x^2-\frac{1}{36}y^2$ by $\frac{2}{3}x+\frac{1}{6}y$.

$$(\tfrac{4}{9}x^2-\tfrac{1}{36}y^2)\div(\tfrac{2}{3}x+\tfrac{1}{6}y)$$
$$=(\tfrac{2}{3}x-\tfrac{1}{6}y)\,(\tfrac{2}{3}x+\tfrac{1}{6}y)\div(\tfrac{2}{3}x+\tfrac{1}{6}y)$$
$$=\tfrac{2}{3}x-\tfrac{1}{6}y.$$

Here we put $a=\frac{2}{3}x$, $b=\frac{1}{6}y$ in the formula $a^2-b^2=(a-b)\,(a+b)$.

Example 3. Divide $x^2-4y^2-16z^2+16yz$ by $x-2y+4z$.

$$\begin{aligned}(x^2-4y^2-16z^2+16yz)&\div(x-2y+4z)\\&=\{x^2-(4y^2-16yz+16z^2)\}\div(x-2y+4z)\\&=\{x^2-(2y-4z)^2\}\div(x-2y+4z)\\&=(x+2y-4z)(x-2y+4z)\div(x-2y+4z)\\&=x+2y-4z.\end{aligned}$$

Here we use the formulæ for $(a-b)^2$ and for a^2-b^2 which we have employed in the two preceding examples.

Example 4. Divide $27x^3y^3-8z^3$ by $3xy-2z$.

$$\begin{aligned}(27x^3y^3-8z^3)&\div(3xy-2z)\\&=(3xy-2z)\{(3xy)^2+(3xy)(2z)+(2z)^2\}\div(3xy-2z)\\&=9x^2y^2+6xyz+4z^2.\end{aligned}$$

Here we use the formula

$$a^3-b^3=(a-b)(a^2+ab+b^2).$$

Example 5. Divide $64x^4+1$ by $8x^2-4x+1$.

$$\begin{aligned}(64x^4+1)&\div(8x^2-4x+1)\\&=(64x^4+16x^2+1-16x^2)\div(8x^2-4x+1)\\&=\{(8x^2+1)^2-(4x)^2\}\div(8x^2+1-4x)\\&=8x^2+1+4x=8x^2+4x+1.\end{aligned}$$

The artifice of adding and subtracting the same quantity is of frequent use in division and resolution into factors.

47. When an expression N is exactly divisible by $x-a$, the meaning is that N is a product which contains $x-a$ for one factor, so that

$$N=Q(x-a),$$

where Q is the quotient which results from dividing N by $x-a$. Thus N vanishes when $x-a$ vanishes, and $x-a$ vanishes only when $x=a$. Hence without the labour of actual division we are provided with a test to determine whether a given expression is exactly divisible by a binomial of the shape of $x-a$ or not; when it is so divisible it vanishes when x is made equal to a.

Also when the given expression N is not exactly divisible by $x-a$ it does not vanish when $x=a$. For in this case divide N by $x-a$ until the remainder R does not contain x and so retains the same value whatever be the value of x, and let Q be the quotient. Then

$$N=Q(x-a)+R;$$

and putting $x=a$, the expression N does not vanish but assumes the value R which is independent of x and is some constant quantity that is not zero.

Example 1. Determine the value of c so that the expression

$$x^4-4x^3+7x^2-2x-c$$

may be exactly divisible by $x-1$.

Substituting 1 for x in the expression, its value

$$=1-4+7-2-c=2-c\,;$$

and this must vanish if the given expression is exactly divisible by $x-1$; hence

$$2-c=0\,;$$

$$\therefore\ c=2.$$

Example 2. Is the expression

$$2x^5-9x^3-85x-6$$

exactly divisible by $x+3$?

Substituting -3 for x, the value of the expression

$$=-486+243+255-6$$

$$=\ 498-492=6\,;$$

thus the value is not zero and the expression is therefore not divisible by $x+3$.

EXAMPLES.

Divide

1. $8a^3$ by $4a$.
2. $12a^4b^3$ by $-3a^2b$.
3. $\frac{2}{3}a^3$ by $\frac{1}{6}a^2$.
4. $-30x^4$ by $6x^2$.
5. $-18x^5y^4z^2$ by $-3x^2y^2z^2$.
6. $7a^2-3a$ by a.
7. $-8a^2+2ab$ by $-2a$.
8. $6x^2-3x$ by $-3x$.
9. $\frac{5}{8}x^2-\frac{3}{4}x$ by $\frac{1}{2}x$.
10. $10xy-15y^2$ by $5y$.
11. $zx-xy-x^2$ by $-x$.
12. a^3-a^2-a by $-a$.
13. $-8a^3+24a^2b-36ab^2$ by $4a$.
14. $12x^3-9x^2+6x$ by $3x$.
15. $48x^3-144x^2-80x$ by $-16x$.
16. a^2+5a+6 by $a+3$.
17. a^2-a-42 by $a-7$.
18. $-21a^2-a+10$ by $7a+5$.
19. $a^2-3ab-28b^2$ by $a-7b$.
20. $x^2-9x+14$ by $x-2$.
21. $x^2+x-132$ by $x+12$.
22. x^2-6x+9 by $x-3$.
23. $-8x^2+10x-3$ by $2x-1$.
24. $14x^2-11x-15$ by $7x+5$.
25. $8x^2+10x+3$ by $2x+1$.

26. $40x^2-x-6$ by $5x-2$.
27. $42x^2-5xy-132y^2$ by $6x-11y$.
28. $24x^2-5xy-36y^2$ by $-8x-9y$.
29. $\frac{5}{9}x^2+\frac{1}{8}x-\frac{1}{8}$ by $\frac{2}{3}x-\frac{1}{4}$.
30. $\frac{1}{2}x^2-\frac{31}{84}xy-\frac{1}{4}y^2$ by $\frac{3}{7}x-\frac{1}{2}y$.
31. $x^3-23x+10$ by x^2-5x+2.
32. $a^3-3a^2-4a+12$ by $a+2$.
33. $42a^3-19a^2b-21ab^2+10b^3$ by $-21a^2-ab+10b^2$.
34. $2x^3-17x^2+38x-15$ by $2x-1$.
35. $4x^3-x^2y-27xy^2-18y^3$ by $x^2-xy-6y^2$.
36. $5x^3-8xy^2+3y^3$ by $x-y$.
37. $625x^4-y^4$ by $5x-y$.
38. $x^4-2x^3+2x^2-1$ by x^3-x^2+x+1.
39. x^5-y^5 by $x-y$.
40. x^6-y^6 by $x+y$.
41. x^6-5x^3-24 by $x-2$.
42. a^3-7a+6 by a^2-3a+2.
43. $10a^3+a^2b-11ab^2+4b^3$ by $5a^2+3ab-4b^2$.
44. $2x^3-x^2-2x+1$ by $2x^2-3x+1$.
45. $2x^4-3x^3y-4x^2y^2+3xy^3+2y^4$ by $x^2-3xy+2y^2$.
46. $24a^4+a^3b-31a^2b^2+26ab^3-2b^4$ by $3a^2-4ab+2b^2$.
47. $x^5-4x^3y^2+4xy^4+8y^5$ by $x^4-2x^3y+4y^4$.
48. $x^6-2x^5-3x^4+6x^3+2x^2-5x+2$ by x^5-3x^3+2x-1.
49. $2x^3-(a+6b+4c)x^2+(12bc+2ca+3ab)x-6abc$ by $2x-a$.
50. $a^2x^4+(ab-ca)x^3y-(bc-ca+ab)x^2y^2-(b^2+c^2)xy^3-bcy^4$ by $ax^2-cxy-by^2$.

Resolve into factors by formulæ and hence divide

51. $25a^2-30ab+9b^2$ by $5a-3b$.
52. $49a^2-4b^2$ by $7a+2b$.
53. $16x^2+72xy+81y^2$ by $4x+9y$.
54. $36x^2-100y^2$ by $6x-10y$.
55. $\frac{1}{4}x^2-\frac{4}{9}y^2$ by $\frac{1}{2}x+\frac{2}{3}y$.
56. $x^3y^3-a^3z^3$ by $xy-az$.
57. $8a^3b^3+1$ by $2ab+1$.
58. $27a^3-64b^3$ by $9a^2+12ab+16b^2$.
59. $125x^3+343y^3$ by $25x^2-35xy+49y^2$.
60. $x^2-y^2-z^2+2yz$ by $x-y+z$.
61. $4a^2-9b^2-25c^2-30bc$ by $2a-3b-5c$.
62. a^4+a^2+1 by a^2+a+1.
63. x^4+2x^2+9 by x^2-2x+3.
64. x^4-8x^2+16 by x^2-4x+4.
65. $x^4+10x^2y^2+49y^4$ by $x^2-2xy+7y^2$.
66. $4x^4+1$ by $2x^2+2x+1$.
67. x^6-y^6 by $x^4+x^2y^2+y^4$.
68. x^2-y^2+2y-1 by $x-y+1$.
69. $4x^4+3x^2y^2+y^4$ by $2x^2-xy+y^2$.

70. $x^{3m}-y^{3m}$ by $x^{2m}+x^my^m+y^{2m}$.

71. $x^{3m}+y^{3m}$ by $x^{2m}-x^my^m+y^{2m}$.

72. $(y-z)^3+(z-x)^3$ by $y-x$.

73. $4x^2-9y^2-z^2+6yz$ by $2x-3y+z$.

74. $(a+b+c)^2-a^2$ by $2a+b+c$.

$(a+b+c)^3-a^3$ by $b+c$.

76. $(2b-c+2a)^3+(2c-b-a)^3$ by $a+b+c$.

77. $a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2$ by $a^2-b^2-c^2-2bc$.

78. $x^4-2x^2y^2+y^4$ by $x-y$.

79. $x^8+x^4y^4+y^8$ by x^2+xy+y^2.

80. $9x^4-7x^2y^2+y^4$ by $3x^2-xy-y^2$.

81. $x^3+y^3+3y^2z+3yz^2+z^3$ by $x+y+z$.

82. $x^3-3x^2y+3xy^2-y^3-z^3$ by $x-y-z$.

83. x^4+4y^4 by $x^2-2xy+2y^2$.

84. $x^6-2x^3y^3+y^6$ by $x^2-2xy+y^2$.

85. $y^2z^2+z^2x^2-x^2y^2-2xyz^2$ by $yz-zx-xy$.

Without actual division shew that the following expressions are exactly divisible:

86. $4x^2-23x-6$ by $x-6$.

87. $6x^2+29x-5$ by $x+5$.

88. $3x^3-14x^2+17x-6$ by $x-3$.

89. $2x^3+x^2-2x+8$ by $x+2$.

90. $2x^4-13x^3+15x^2-2x+10$ by $x-5$.

91. $4x^4+2x^3-x^2-2x-3$ by $x+1$.

Without actual division ascertain whether the following expressions are exactly divisible:

92. $5x^2-28x-18$ by $x-6$.

93. $4x^2+13x-8$ by $x+4$.

94. $3x^3-5x^2-x-2$ by $x-2$.

95. $2x^3+5x^2-4x+3$ by $x+3$.

96. $7x^4+3x^3-4x^2-3x-2$ by $x-1$ or $x+1$.

Find for what value of a the following expressions are exactly divisible:

97. ax^3-8x^2+7x-4 by $x-1$.

98. $3x^3+ax^2-13x+35$ by $x+5$.

99. $4x^4-13x^3-17x^2+ax+12$ by $x-4$.

100. $6x^5+5x^4-10x^3+5x^2-15x+a$ by $x+2$.

CHAPTER VI.

FACTORS.

48. To resolve an expression into its component factors is the same as to find the expressions which will divide it, and may frequently be effected by the use of formulæ such as those of Art. 31—33 or by rules such as those of Art. 45, the particular formula or rule which is to be applied being suggested by the shape of the given expression. When no formula is immediately applicable the expression may often be put into a suitable form by re-arranging the terms or by adding and subtracting the same quantity or by resolving one term or more terms into component parts. The re-arrangement of terms is especially useful in the treatment of identities and other problems which involve symmetrical products of three letters; and in examples of that kind, where the result is given, valuable information is conveyed to the student which will enable him to arrange the terms in the order most convenient for his purpose.

In the preceding two chapters we have passed many instances of resolution into factors by means of formulæ or of the converse operation. The examples 41—65 of Chap. IV. and 51—85 of Chap. V. deal respectively with the formation of products and with resolution into factors by formulæ. An expression can at once be resolved into factors if we can by transformation write it in the shape of *the difference of two squares* or of *the sum or difference of two cubes;* and the desired transformation is generally effected by changing the order of the terms or by substituting for one or more of the terms the sum or difference of new terms suitably chosen for the purpose.

Example 1. x^4-11x^2+1

$=x^4-2x^2+1-9x^2$

$=(x^2-1)^2-(3x)^2$

$=(x^2-1-3x)(x^2-1+3x)$

$=(x^2-3x-1)(x^2+3x-1).$

Example 2. $(2a-b)^3+(2b-a)^3$

$=(2a-b+2b-a)\{(2a-b)^2-(2a-b)(2b-a)+(2b-a)^2\}$

$=(a+b)(7a^2-13ab+7b^2)$.

Example 3. $x^3(x+2y)-y^3(y+2x)$

$=x^4-y^4+2xy(x^2-y^2)$

$=(x^2-y^2)(x^2+y^2+2xy)$

$=(x^2-y^2)(x+y)^2$

$=(x-y)(x+y)^3$.

Example 4. $b^2+c^2+2bc-2ca-2ab$

$=a^2+b^2+c^2+2bc-2ca-2ab-a^2$

$=(b+c-a)^2-a^2$

$=(b+c)(b+c-2a)$.

Example 5. Prove the identity

$$(y-z)^3+(z-x)^3+(x-y)^3=3(y-z)(z-x)(x-y).$$

Here $(y-z)^3+(z-x)^3+(x-y)^3$

$=(y-z)^3+z^3-3xz^2+3x^2z-x^3+x^3-3x^2y+3xy^2-y^3$

$=(y-z)^3-(y^3-z^3)+3x(y^2-z^2)-3x^2(y-z)$

$=(y-z)\{(y-z)^2-(y^2+yz+z^2)+3x(y+z)-3x^2\}$

$=(y-z)\{y^2-2yz+z^2-y^2-yz-z^2+3xy+3zx-3x^2\}$

$=(y-z)\{3zx-3x^2-3yz+3xy\}$

$=3(y-z)\{x(z-x)-y(z-x)\}$

$=3(y-z)(z-x)(x-y)$.

In this example the solution principally depends on our re-arranging the terms and placing them in such an order as to make the common factor apparent.

An **identity** is a relation which is true for all values of the letters; thus $x^2-y^2=(x-y)(x+y)$ and $(a-b)^2=a^2-2ab+b^2$ are identities.

49. A class of problems on resolution into factors which may be dealt with according to definite rules is that in which the expression to be resolved is of the form x^2+dx+e; such an expression containing powers of x up to the square but not

beyond is called a **quadratic expression**. A consideration of the identities

$$(x-a)(x-b) = x^2-(a+b)x+ab,$$
$$(x+a)(x+b) = x^2+(a+b)x+ab,$$
$$(x+a)(x-b) = x^2+(a-b)x-ab,$$
$$(x-a)(x+b) = x^2-(a-b)x-ab,$$

shews that if a trinomial quadratic expression in x, such as x^2+dx+e, is capable of being resolved into component binomial factors of the first degree, then *the coefficient of x is the algebraical sum of the second terms of the binomials and the term independent of x is the product of the second terms.*

We also notice that when e is positive the second terms of the binomial factors are both positive if d be positive and both negative if d be negative; and that when e is negative the second terms are of opposite signs and d has the sign of the numerically greater term.

Thus we have

$$x^2-7x+10 = (x-2)(x-5),$$
$$x^2+7x+10 = (x+2)(x+5);$$

here we are to determine two numbers of the same sign whose algebraical sum is in one case -7 and in the other 7, i.e. whose arithmetical sum is in both cases 7, and whose product is 10, and by trial we see that 2 and 5 alone satisfy these conditions.

Again we have

$$x^2+x-12 = (x-3)(x+4),$$
$$x^2-x-12 = (x+3)(x-4);$$

and here we are to determine two numbers of opposite signs whose algebraical sum is in one case 1 and in the other -1 and whose product is -12, i.e. whose arithmetical difference is in both cases 1 and product 12, and by trial we find these numbers to be 3 and 4.

The solution of this class of problems thus depends entirely on our ability to discover two numbers in arithmetic whose product and whose sum or difference are given.

An alternative method is by inspection to replace some one term by the sum or difference of two new terms chosen suitably for our purpose. Thus

$x^2-5x+4 = x^2-x-4x+4 = x(x-1)-4(x-1) = (x-1)(x-4)$;

$x^2+6x-16 = x^2-4+6x-12 = (x-2)(x+2)+6(x-2) = (x-2)(x+8)$.

50. The resolution into factors of a quadratic expression in which the coefficient of x^2 is not unity is a little more difficult, but can be reduced to the foregoing case in which the coefficient of x^2 is unity by multiplying and dividing the whole expression by the coefficient of x^2 and by substitution. Thus if the given expression be $cx^2 + dx + e$ it may be written in the form

$$\frac{1}{c}(c^2x^2 + cdx + ce),$$

i.e. in the form $\frac{1}{c}(y^2 + dy + ce)$, where $cx = y$; and $y^2 + dy + ce$ is of the typical form considered in the last Article and can be resolved into its component factors by the method there explained.

Thus
$$\begin{aligned} 6x^2 - 11x + 4 &= \tfrac{1}{6}(36x^2 - 66x + 24) \\ &= \tfrac{1}{6}(y^2 - 11y + 24), \text{ where } y = 6x, \\ &= \tfrac{1}{6}(y - 3)(y - 8) \text{ by the ordinary rule,} \\ &= \tfrac{1}{3}(6x - 3) \times \tfrac{1}{2}(6x - 8) \\ &= (2x - 1)(3x - 4). \end{aligned}$$

In an example like this we multiply and divide the whole expression by the coefficient of x^2 so as to make the term in x^2 a complete square, under which circumstances it may be replaced by y^2.

Or, as in the last Article, an alternative method is to resolve some one term into the sum or difference of two new terms so selected as to make one factor apparent. Thus

$$\begin{aligned} 35x^2 - x - 12 &= 35x^2 - 21x + 20x - 12 \\ &= 7x(5x - 3) + 4(5x - 3) = (5x - 3)(7x + 4); \\ 4x^2 - 11x + 6 &= 4x^2 - 16 - 11x + 22 \\ &= 4(x^2 - 4) - 11(x - 2) = (x - 2)\{4(x + 2) - 11\} \\ &= (x - 2)(4x - 3). \end{aligned}$$

This method however requires considerable expertness, and the student is recommended to rely principally on the general rule explained in the earlier part of the Article.

The investigation of this and of the preceding Article also applies to all expressions of a quadratic form which involve letters symmetrically.

Thus by the processes just explained we have

$$a^2 - 4ab - 21b^2 = (a + 3b)(a - 7b);$$
$$x^2 - 5ax + 4a^2 = (x - a)(x - 4a);$$
$$3y^2z^2 + 2xyz - 8x^2 = (yz + 2x)(3yz - 4x);$$
$$x^4 - 7x^2 - 18 = (x^2 + 2)(x^2 - 9).$$

51. An expression involving powers of x up to the cube but not beyond is called a **cubic expression.**

A consideration of the identity

$$(x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (bc + ca + ab)x - abc$$

at once shews that a cubic expression of the form $x^3 + dx^2 + ex + f$ is capable of resolution into factors by laws similar to those which hold for quadratic expressions; that the sum of the second terms of the binomial factors is the coefficient of x^2, that the sum of the products of these terms taken two and two is the coefficient of x, and that the product of these terms is the term in the expression which is independent of x. Also by Art. 47 if $x - a$ be a factor of an expression we know that the expression will vanish when $x = a$. Hence when required to resolve into factors a cubic expression $x^3 + dx^2 + ex + f$, we know that the term f consists of factors which are the second terms of the binomials and so need only try such numbers as are factors of f; if a be one such factor, then by substituting a for x in the expression we at once discover whether $x - a$ is a factor by noticing whether the expression then vanishes or not. If $x - a$ is found to be a factor, then the remaining factor is a quadratic expression which may be treated by the methods of Art. 49. Also cubic expressions and expressions of a higher degree may frequently be resolved into their component factors by splitting one or more of the terms into new terms so chosen as to make a component factor apparent.

Example 1. Resolve $x^3 + 2x^2 - 21x + 18$ into factors.

The only factors of 18 are 1, 2, 3, 6, 9, 18, and we find on trial that the expression vanishes when $x = 1$ or 3 or -6, so that the factors are $x - 1$, $x - 3$, $x + 6$.

Example 2. $x^3 + 2x^2 - 3 = x^3 - 1 + 2(x^2 - 1)$

$$= (x - 1)(x^2 + x + 1) + 2(x - 1)(x + 1) = (x - 1)(x^2 + 3x + 3);$$

or $= x^3 - x^2 + 3(x^2 - 1) = x^2(x - 1) + 3(x - 1)(x + 1) = (x - 1)(x^2 + 3x + 3)$.

In this example again the only factors of 3 are 1 and 3 and by trial we see that the given expression vanishes when $x = 1$, so that $x - 1$ is known to be a factor, and by division the remaining quadratic factor may be obtained.

EXAMPLES.

Resolve into their component factors :

1. a^2+a-b^2-b.
2. $ac+bd+ad+bc$.
3. $ac+bd-ad-bc$.
4. $a^2-2ab+2bc-c^2$.
5. $b(a-b)+c(c-a)$.
6. $a^2+b^2+2bc+2ca+2ab$.
7. $a^3-a^2b-ab^2+b^3$.
8. $a^3-a^2b+ab^2-b^3$.
9. $c^2(a-b)+b^2(c-a)$.
10. $a^3-3a^2b+3ab^2-b^3-1$.
11. $a^3-2a^2b+2ab^2-b^3$.
12. $(a-b)(b+x)+(c-a)(c+x)$.
13. $a^3+(m+1)(a^2b+ab^2)+b^3$.
14. $ma^3+na^2b-mab^2-nb^3$.
15. $a^2+b^2-c^2-d^2+2ab-2cd$.
16. $a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2$.
17. $x^2-9x+14$.
18. $x^2-5x-36$.
19. $x^2+16x+63$.
20. $x^2-10x+24$.
21. $x^2-xy-156y^2$.
22. $x^2+3ax-88a^2$.
23. $x^2+2ax+a^2-b^2$.
24. $x^2-2bx-a^2+b^2$.
25. $4x^2+x-14$.
26. $3x^2+7x-40$.
27. $2x^2-3x-54$.
28. $12x^2+xy-20y^2$.
29. $abx^2-(a^2+b^2)x+ab$.
30. $a^2bx^2-ax-b+1$.
31. $c^2x^2-acx+ab-b^2$.
32. $(m^2-1)x^2-2mnxy+n^2y^2$.
33. $x^4-2x^2yz-35y^2z^2$.
34. $x^3-13x+12$.
35. x^4-x^2+2x-1.
36. $x^3-x^2-8x+12$.
37. x^4-18x^2+1.
38. x^4+3x^2+4.
39. $x^4+2x^2y^2+9y^4$.
40. $abx^3+(a^2-b)x^2-1$.

Demonstrate the following identities :

41. $bc(b-c)+ca(c-a)+ab(a-b)=-(b-c)(c-a)(a-b)$.
42. $a^2(b-c)+b^2(c-a)+c^2(a-b)=-(b-c)(c-a)(a-b)$.
43. $(b-c)(x-b)(x-c)+(c-a)(x-c)(x-a)+(a-b)(x-a)(x-b)$
$=-(b-c)(c-a)(a-b)$.
44. $(b-c)(a-x)(a+x)+(c-a)(b-x)(b+x)+(a-b)(c-x)(c+x)$
$=-(b-c)(c-a)(a-b)$.
45. $(a+b+c)(bc+ca+ab)-abc=(b+c)(c+a)(a+b)$.
46. $(x+y+z)^3-(x^3+y^3+z^3)=3(y+z)(z+x)(x+y)$.

47. $x^3(y-z)+y^3(z-x)+z^3(x-y)=-(y-z)(z-x)(x-y)(x+y+z)$.

48. $x(y-z)^3+y(z-x)^3+z(x-y)^3=(y-z)(z-x)(x-y)(x+y+z)$.

49. Deduce the result of Ex. 41 from Ex. 43 and the result of Ex. 42 from Ex. 44.

50. Resolve the expression

$$x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)$$

into its simplest factors and hence shew that it is equal to

$$a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2),$$

where $x-a=y-b=z-c$.

Additional examples on resolution into factors are Ex. 11—60 Chap. VII. and Ex. 11—60 Chap. VIII.

CHAPTER VII.

HIGHEST COMMON FACTOR.

52. The **Highest Common Factor** of two or more algebraical expressions is the expression of highest dimensions which will divide each of them exactly. The name formerly in use and borrowed from arithmetic was Greatest Common Measure, which however is inappropriate and liable to convey the false impression that we are comparing magnitudes and that the numerical values of algebraical expressions are under consideration. The highest common factor is frequently written in the abbreviated form H.C.F.

The H.C.F. of simple expressions follows immediately from the definition and can always be determined by inspection. It consists of *the* G.C.M. *of the numerical coefficients prefixed to the product of the highest power of each letter which will divide each of the given expressions exactly.*

Thus the H.C.F. of a^4b^2 and a^3b^3 is a^3b^2, and of $12x^2y^2z^2$, $9x^3y^2z$, $18x^4y^3z^3$ is $3x^2y^2z$. For in the first example the highest power of a that will divide a^3 and a^4 is a^3 and of b that will divide b^2 and b^3 is b^2; and in the second example the greatest number that will divide 9, 12 and 18 is 3, the highest power of x that will divide x^2, x^3 and x^4 is x^2, of y that will divide y^2 and y^3 is y^2 and of z that will divide z, z^2 and z^3 is z.

53. The H.C.F. of compound expressions can often, but not always, be determined by inspection. Its discovery depends on our ability to resolve the expressions under treatment into their component factors; whenever this can be effected by formulæ or rules or methods such as those suggested in the last chapter, the H.C.F. can at once be written down from the definition. It consists of *the* G.C.M. *of the numerical coefficients prefixed to the product of each common factor raised to the highest power which will divide each of the given expressions exactly.*

Thus the expressions $x^2-7x+10$ and x^2+x-6 are equivalent to $(x-2)(x-5)$ and $(x-2)(x+3)$ respectively and their H.C.F. is by inspection $x-2$.

Again the expressions

$$2(x^3+a^3),\ 6x(x^2-a^2) \text{ and } 4(x+a)(x^2-3ax-4a^2)$$

are equivalent to

$$2(x+a)(x^2-ax+a^2),\ 6x(x+a)(x-a) \text{ and } 4(x+a)^2(x-4a)$$

respectively, and their H.C.F. is at once seen to be $2(x+a)$.

Whatever be the shape of the given expressions, if they contain any simple factors, these factors should first be removed and their H.C.F. found by inspection as in the foregoing example, and we must investigate separately the H.C.F. of the expressions which remain after removal of the simple factors.

It appears then on the whole that the only expressions of which we do not know how to find the H.C.F. are expressions which do not readily admit of resolution into their compound factors and which do not contain any simple factor.

54. Let A and B denote two expressions having no simple factor and arranged according to descending powers of some common letter, and let the term of highest degree in that letter in A be not lower than that of highest degree in B.

Divide A by B and let a be the quotient and C the remainder, so that

$$A=aB+C.$$

Now any common factor of B and C must obviously be a factor of $aB+C$, i.e. of A, and thus being already by hypothesis a factor of B it is proved to be also a factor of A and so must be a common factor of A and B.

Again writing the equation in the form

$$C=A-aB$$

we see that any common factor of A and B must be a factor of $A-aB$, i.e. of C, and thus being already by hypothesis a factor of B it is proved to be also a factor of C and so must be a common factor of B and C.

We thus arrive at the conclusion that every common factor of B and C is a common factor of A and B, and therefore the H.C.F. of B and C is the H.C.F. of A and B.

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Again divide B by C and let b be the quotient and D the remainder so that

$$B = bC + D,$$

and therefore also $$D = B - bC.$$

Here every common factor of C and D must be a factor of $bC + D$, i.e. of B, and every common factor of B and C must be a factor of $B - bC$, i.e. of D; hence by the same argument as that used above every common factor of C and D is a common factor of B and C, i.e. is a common factor of A and B. Therefore the H.C.F. of C and D is the H.C.F. of B and C, i.e. is the H.C.F. of A and B; and so on. This process may be continued until there is no remainder; in all such divisions every factor of the divisor and dividend is a factor of the original divisor and dividend, and the H.C.F. of every divisor and dividend is thus always the H.C.F. of the original expressions A and B. When the process has been continued so far that there is no remainder, the H.C.F. of the last divisor and dividend is still the H.C.F. of A and B; but the H.C.F. of the last divisor and dividend is plainly the divisor itself, and this divisor is therefore the H.C.F. required.

If at any stage of this process a *simple* factor occur in divisor or dividend it may be removed, and similarly at any stage a simple factor may be introduced in the divisor or dividend without affecting the result; for the process applies only to *compound* factors, which, so far as their H.C.F. is concerned, remain unaffected by the removal or introduction of a *simple* factor. This is at once obvious by considering any example that may occur; for instance, the H.C.F. of $x^2 - 3x + 2$ and $5x^2 - 7x - 6$ is $x - 2$ and is obviously the same as the H.C.F. of $3x(x^2 - 3x + 2)$ and $5x^2 - 7x - 6$ or of $x^2 - 3x + 2$ and $2(5x^2 - 7x - 6)$, and so on.

As we perform the successive divisions each remainder is necessarily of lower dimensions than the preceding from the very nature of division; hence as the operations are continued we must at last arrive at a stage when there is no remainder, or at a stage when there is a remainder that does not involve the common letter if we stop short a little earlier. In this latter case the given expressions have no common factor except unity; and, having no common compound factor, are generally described as having no H.C.F.

55. We are now in a position to state the rule for finding the H.C.F. of two expressions, neither of which has any simple factor:

Arrange the two expressions according to descending powers of some common letter and divide the expression of highest degree in that letter by the other; or if they are of the same degree, divide either by the other. Make the remainder a new divisor and the former divisor a new dividend, and continue the process until there is no remainder. The last divisor will be the H.C.F. *required.*

Example 1. Find the H.C.F. of

$$2x^3+7x^2-9x-26 \text{ and } 2x^4+9x^3-36x-32.$$

$$\begin{array}{r} 2x^3+7x^2-9x-26 \;)\; 2x^4+9x^3 \quad -36x-32 \;(\; x+1 \\ \underline{2x^4+7x^3-9x^2-26x} \\ 2x^3+9x^2-10x-32 \\ \underline{2x^3+7x^2-\;9x-26} \\ 2x^2-\;\;x-\;6 \end{array}$$

$$\begin{array}{r} 2x^2-x-6 \;)\; 2x^3+7x^2-9x-26 \;(\; x+4 \\ \underline{2x^3-\;x^2-6x} \\ 8x^2-3x-26 \\ \underline{8x^2-4x-24} \\ x-\;2 \end{array}$$

$$\begin{array}{r} x-2 \;)\; 2x^2-\;x-6 \;(\; 2x+3 \\ \underline{2x^2-4x} \\ 3x-6 \\ \underline{3x-6} \end{array}$$

The process is here carried on in accordance with the rule just enunciated and the last divisor $x-2$ is the H.C.F. required.

Example 2. Find the H.C.F. of

$$x^3+x^2-17x+15 \text{ and } x^3+2x^2-19x-20.$$

$$\begin{array}{r} x^3+x^2-17x+15 \;)\; x^3+2x^2-19x-20 \;(\; 1 \\ \underline{x^3+\;x^2-17x+15} \\ x^2-\;2x-35 \end{array}$$

$$\begin{array}{r} x^2-2x-35 \;)\; x^3+\;x^2-17x+15 \;(\; x+3 \\ \underline{x^3-2x^2-35x} \\ 3x^2+18x+\;15 \\ \underline{3x^2-\;6x-105} \\ 24 \;)\; 24x+120 \\ \underline{\quad\quad x+\;\;5} \end{array}$$

$$\begin{array}{r} x+5 \;)\; x^2-2x-35 \;(x-7 \\ \underline{x^2+5x} \\ -7x-35 \\ \underline{-7x-35} \end{array}$$

In this example the highest power of x is the same in the two expressions and it is immaterial which of them is made the divisor. When we arrive at the remainder $24x+120$ we notice that 24 is a common factor and so may be removed in accordance with the principle proved in the last Article. Accordingly we divide by this factor and so reject it, and we make $x+5$ the new divisor and not $24x+120$. The last divisor $x+5$ is the H.C.F. required.

Example 3. Find the H.C.F. of

$$x^3-2ax^2-9a^2x+18a^3 \text{ and } x^4+4ax^3-11a^2x^2-66a^3x-72a^4.$$

$$\begin{array}{l}
x^3-2ax^2-9a^2x+18a^3 \;)\; x^4+4ax^3-11a^2x^2-66a^3x-72a^4 \;(\; x+6a \\
\qquad\qquad x^4-2ax^3-9a^2x^2+18a^3x \\
\qquad\qquad\qquad 6ax^3-2a^2x^2-84a^3x-72a^4 \\
\qquad\qquad\qquad 6ax^3-12a^2x^2-54a^3x+108a^4 \\
\qquad\qquad\qquad 10a^2 \;)\; 10a^2x^2-30a^3x-180a^4 \\
\qquad\qquad\qquad\qquad\qquad x^2-3ax-18a^2
\end{array}$$

$$\begin{array}{l}
x^2-3ax-18a^2 \;)\; x^3-2ax^2-9a^2x+18a^3 \;(\; x+a \\
\qquad\qquad x^3-3ax^2-18a^2x \\
\qquad\qquad\qquad ax^2+9a^2x+18a^3 \\
\qquad\qquad\qquad ax^2-3a^2x-18a^3 \\
\qquad\qquad\qquad 12a^2 \;)\; 12a^2x+36a^2 \\
\qquad\qquad\qquad\qquad\qquad x+3a
\end{array}$$

$$\begin{array}{l}
x+3a \;)\; x^2-3ax-18a^2 \;(\; x-6a \\
\qquad\qquad x^2+3ax \\
\qquad\qquad\qquad -6ax-18a^2 \\
\qquad\qquad\qquad -6ax-18a^2
\end{array}$$

Here in accordance with the principle proved in the last Article we remove the simple factors $10a^2$, $12a^2$, from the first and second remainders respectively before we make them the new divisors. The last divisor $x+3a$ is the H.C.F. required.

Example 4. Find the H.C.F. of

$$3a^3-10a^2b+ab^2+6b^3 \text{ and } a^4-2a^3b-4a^2b^2+2ab^3+3b^4.$$

$$\begin{array}{l}
3a^3-10a^2b+ab^2+6b^3 \;)\; a^4-2a^3b-4a^2b^2+2ab^3+3b^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3 \\
\qquad\qquad 3a^4-6a^3b-12a^2b^2+6ab^3+9b^4 \;(\; a \\
\qquad\qquad 3a^4-10a^3b+a^2b^2+6ab^3 \\
\qquad\qquad\qquad 4a^3b-13a^2b^2+9b^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3 \\
\qquad\qquad\qquad 12a^3b-39a^2b^2+27b^4 \;(\; 4b \\
\qquad\qquad\qquad 12a^3b-40a^2b^2+4ab^3+24b^4 \\
\qquad\qquad\qquad\qquad b^2 \;)\; a^2b^2-4ab^3+3b^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad a^2-4ab+3b^2
\end{array}$$

$$\begin{array}{l} a^2-4ab+3b^2 \;)\; 3a^3-10a^2b+\;ab^2+6b^3 \;(\; 3a+2b \\ \qquad\quad 3a^3-12a^2b+9ab^2 \\ \qquad\quad \overline{\qquad\qquad 2a^2b-8ab^2+6b^3} \\ \qquad\quad 2a^2b-8ab^2+6b^3 \\ \qquad\quad \overline{\qquad\qquad\qquad\qquad} \end{array}$$

Here in accordance with the principle which permits us to introduce or remove a factor when convenient, we multiply the original dividend and first remainder by 3 and from the next remainder we reject the factor b^2 before making this remainder the next divisor. The last divisor $a^2-4ab+3b^2$ is the H.C.F. required.

Example 5. Find what numerical values a can have so that the expressions x^2-x-2a and $x^2-3x-3a$ may contain a common factor.

The condition that these expressions may contain a common factor is plainly that in the process of finding their H.C.F. the last remainder shall vanish. In accordance with the rule we have the following work:

$$\begin{array}{l} x^2-3x-3a \;)\; x^2-\;x-2a \;(\; 1 \\ \qquad\quad x^2-3x-3a \\ \qquad\quad \overline{\qquad\qquad 2x+\;a} \end{array}$$

$$\begin{array}{l} 2x+a \;)\; x^2-3x-3a \\ \qquad\qquad\qquad\quad 2 \\ \qquad\quad \overline{2x^2-6x-6a} \;(\; x-(3+\tfrac{1}{2}a \\ \qquad\quad 2x^2+ax \\ \qquad\quad \overline{-(6+a)x-6a} \\ \qquad\quad -(6+a)x-3a\;-\tfrac{1}{2}a^2 \\ \qquad\quad \overline{\qquad\qquad\qquad\quad \tfrac{1}{2}a^2-3a} \end{array}$$

Hence we must have $\frac{1}{2}a^2-3a=0$, i.e. $a(\frac{1}{2}a-3)=0$; therefore $a=0$; or $\frac{1}{2}a-3=0$, i.e. $\frac{1}{2}a=3$, $\therefore\ a=6$. If a have the value zero the expressions take the form $x(x-1)$, $x(x-3)$, and contain the common factor x; if a have the value 6 the expressions take the form $(x+3)(x-4)$, $(x+3)(x-6)$, and contain the common factor $x+3$.

56. It is manifest from the definition of the H.C.F. of two expressions that every common factor of the two expressions is a factor of their H.C.F.

It is also obvious that every common factor of two expressions must be a factor of their sum and difference and of the sum and difference of the expressions multiplied by any factors. For let A and B be the two expressions, and let them have the common factor C so that $A=aC$, $B=bC$; then $A+B=(a+b)C$, $A-B=(a-b)C$, $mA+nB=(ma+nb)C$, $mA-nB=(ma-nb)C$; hence C is a factor of $A+B$, $A-B$, $ma+nB$, $mA-nB$, the other factor being $a+b$, $a-b$, $ma+nb$, $ma-nb$ respectively. These

considerations enabled us to obtain the result of Art. 54 and are frequently of use in the solution of problems.

Example 1. Find the common factor of the first degree of the two expressions x^3+2x^2-7x+4 and $x^3+2x^2-11x-12$.

The difference of the expressions

$$=x^3+2x^2-7x+4-(x^3+2x^2-11x-12)$$
$$=4x+16=4(x+4).$$

Here we have given that the expressions have a common factor of the first degree and we know that their common factor must also be a factor of their difference; therefore the common factor is $x+4$.

Example 2. Shew that the two expressions $2x^4+x^3+x^2+x+2$ and $x^4+2x^3+2x^2+2x+1$ have no common factor in x.

Any common factor of $2x^4+x^3+x^2+x+2$ and $x^4+2x^3+2x^2+2x+1$ is a factor of

$$2(2x^4+x^3+x^2+x+2)-(x^4+2x^3+2x^2+2x+1)$$

and of $\quad 2(x^4+2x^3+2x^2+2x+1)-(2x^4+x^3+x^2+x+2)$,

i.e. is a factor of $3(x^4+1)$ and of $3x(x^2+x+1)$; and these last two expressions have by inspection no common factor in x, so that the original two expressions have also no common factor in x.

In this class of problems it is more frequently of service to subtract than to add, and the factors employed should generally be numerical and so chosen as to make some of the terms cancel each other on addition or subtraction.

57. We have already seen in Art. 53 that the problem of finding the H.C.F. of any number of expressions presents no difficulty if the expressions can readily be resolved into their component factors. Thus the only case which remains to be considered is when there are more than two expressions and the factors of the expressions cannot be discovered by inspection.

Now from the nature of the H.C.F. it is obvious that every common factor of three or more expressions must be a factor of the H.C.F. of any two of the expressions. From this consideration it follows that *we must first find the* H.C.F. *of two of the expressions, then the* H.C.F. *of this* H.C.F. *and of a third expression, and so on. The last* H.C.F. *so found is the* H.C.F. *of all the expressions.*

Example. Find the H.C.F. of

$3x^3-10x^2+x+6$, $\quad x^4-2x^3-4x^2+2x+3$ and $\quad x^4+2x^3-17x^2-18x+72$.

Proceeding by the ordinary rule the H.C.F. of $3x^3-10x^2+x+6$ and $x^4-2x^3-4x^2+2x+3$ is found to be x^2-4x+3. And by the same rule the H.C.F. of x^2-4x+3 and $x^4+2x^3-17x^2-18x+72$ is found to be $x-3$, which is therefore the H.C.F. required.

EXAMPLES.

Write down the H.C.F. of

1. a^2b^4, a^3b^2.
2. $3a^3b^2c$, $6a^4b^3c^2$.
3. $2ab^4c^3$, $8a^3b^2c^4$.
4. $5x^2y$, $15x^3y^2$.
5. $13x^2y^3z^4$, $52x^3y^2z$.
6. $3x^3y^2z^2$, $12x^2y^3z^2$, $9x^2y^2z^3$.
7. $6a^2x$, $4a^3y^2$, $8axy$.
8. $19ab^2c$, $57a^2b^3c^2$, $76a^3bc^3$.
9. $23a^2b^3c^3$, $46a^3b^3c^3$, $92a^4b^2c^2$, $a^2b^2c^4$.
10. $48x^4y^3z$, $144x^2y^2z^3$, $16x^3y^3z^2$, $40y^2z^4$.

By resolution into factors find the H.C.F. of

11. a^2-b^2, a^3+b^3.
12. $9(a+b)^2$, $24(a^2-b^2)$.
13. $4a^4+4a^3+4a^2$, $6a^4-6a$.
14. a^3-b^3, $(a+b)(a^2+ab+b^2)$.
15. $8a^6b^3-8a^2b^7$, $48a^7b^2+48ab^8$.
16. $ac-bd+bc-ad$, $ac+bd+bc+ad$.
17. $ac+bd-bc-ad$, $ac-bd-bc+ad$.
18. $ac-bd+bc-ad$, c^2-d^2.
19. $a^2-b^2+ac-bc$, $a^2-b^2-c^2-2bc$.
20. $a^2+b^2-c^2+2ab$, $a^2-b^2-c^2+2bc$.
21. $b^2+c^2+2bc+2ca+2ab$, $2a^2+ca+ab$.
22. $a^3+a^2b+ab^2+b^3$, $a^3-a^2b+ab^2-b^3$.
23. $a^3-3a^2b+3ab^2-b^3$, a^2b-ab^2.
24. $a^4+a^2b^2+b^4$, $(a+b)^4-a^2b^2$.
25. $a^4+b^4+c^4-2b^2c^2-2c^2a^2+2a^2b^2$, $a^4-b^4-c^4+2b^2c^2$.
26. x^2-5x+6, x^2+x-12.
27. x^2+4x+3, x^2-7x-8.
28. $x^2+9x-36$, $x^2+2x-120$.
29. $x^2-11x+30$, $x^2-14x+45$.
30. $x^2-12xy-13y^2$, $x^2+12xy+11y^2$.
31. $x^2-13ax-30a^2$, $x^2-11ax-60a^2$.
32. $2x^2+8x-42$, $4x^3+44x^2+112x$.
33. $3x^2+7x-6$, $3x^2-8x+6$.
34. $7x^2-4x-3$, $14x^2-x-3$.
35. x^2-x-56, $2x^2+9x-35$.
36. $3x^2+2x-21$, $6x^2+x-35$.
37. $14x^2+3x-2$, $14x^2-11x+2$.

38. $8x^2-10xy-3y^2$, $10x^2-11xy-6y^2$.

39. $12x^2-xy-y^2$, $9x^2+12xy-5y^2$.

40. $x^2-7ax+6a^2$, $x^2+ax-42a^2$.

41. $a^2x^2-4axy-32y^2$, $a^2x^2-7axy-8y^2$.

42. $6x^3-27x^2-15x$, $12x^4-42x^3-24x^2$.

43. x^2-4x+3, x^3-4x^2+3.

44. x^3-4x+3, x^4-4x^2+3.

45. x^4+4x^2-12, x^4-x^2-2.

46. x^4-14x^2+48, x^4-64.

47. x^4-7x^2+1, $x^4-3x^3+x^2$.

48. x^2-4x+3, x^2-x-6, x^2+x-12.

49. $x^2-7x+10$, $x^2+2x-35$, x^2+x-30.

50. $4x^2-9x+2$, $3x^2-4x-4$, $2x^2-3x-2$.

51. $3a^2x^2-2abx-8b^2$, $a^2x^2+2abx-8b^2$, $a^3x^3-8b^3$.

52. x^2-9a^2, $x^2+ax-12a^2$, x^3-27a^3.

53. $3x^2-2x-1$, x^3-x^2-x+1, x^3-x^2+x-1.

54. $x^4+x^2y^2+y^4$, $x^5+x^2y^3$, $x^3y^2+y^5$.

55. $3(x^3y^2-x^2y^3)$, $6(x^4y^2-x^2y^4)$, $9(x^5y^2-x^2y^5)$, $12(x^6y^2-x^2y^6)$.

56. $x^2-(a+b)x+ab$, $x^2-(a-b)x-ab$, $x^2+(c-a)x-ca$.

57. $x^2+(a-b)x-ab$, $x^2+(a-c)x-ca$,
$x^3+(a-b-c)x^2+(bc-ca-ab)x+abc$.

58. $(x+y+z)^2(x+y-z)$, $(x+y)(x^2+y^2-z^2+2xy)$, $4y^2z^2-(x^2-y^2-z^2)^2$.

59. $x^5y+x^3y^3+xy^5$, $(x^2y+xy^2)(x^3-y^3)$, $xy(x^2+xy+y^2)^2$.

60. x^4-y^4, x^6-y^6, x^8-y^8, $x^{12}-y^{12}$.

Find the H.C.F. of

61. $a^3-2a^2b-7ab^2-4b^3$, $a^3+4a^2b-ab^2-4b^3$.

62. $a^3-ab^2-6b^3$, $a^3+a^2b-5ab^2-2b^3$.

63. $a^4+2a^3+a^2-2a-2$, $a^4-2a^3+3a^2-4a+2$.

64. $3a^3-2a^2b+3ab^2-2b^3$, $2a^4-a^2b^2-3b^4$.

65. $x^3-13x-12$, x^3+x^2-5x+3.

66. x^3+3x^2-4, $x^4-2x^3-11x^2-12x+36$.

67. $x^3+x^2-16x+20$, $x^3-x^2-22x+40$.

68. x^3-8x-3, $x^4+2x^3-x^2+2x+1$.

69. x^4+3x^2-2x+3, $x^4+2x^3-x^2+2x+1$.

70. $x^4-x^3-6x^2+4x+8$, $x^4+3x^3-3x^2-12x-4$.

71. $x^4+2ax^3-2a^2x^2-6ax^3-3a^4$, $x^4+2a^2x^2-3a^4$.

72. $2x^3+x^2-2x-6$, $2x^3+x^2-8x+3$.

73. $2x^3-x+1$, $3x^3+x^2-x+1$.

74. x^3+x^2-4x-4, $3x^3-8x^2+3x+2$.

75. $3x^3+5x^2y+xy^2-y^3$, $4x^3+3x^2y+y^3$.

76. $2x^3-5x^2-2x-3$, $5x^3-15x^2+3x-9$.

77. $6x^4-42x^2+36x$, $3x^5+15x^4-39x^3+21x^2$.

78. $2x^4+2x^3-32x^2+40x$, $2x^4+x^3-7x^2-4x-4$.

79. $x^4+x^3y-9x^2y^2+11xy^3-4y^4$, $x^4-x^3y-3x^2y^2+5xy^3-2y^4$.

80. $3x^4+x^3y-3xy^3-y^4$, $3x^4-8x^3y+6x^2y^2-y^4$.

81. x^3-2x^2-x+2, x^3-x^2-4, x^3-x^2+x-6.

82. $x^3-x^2y+xy^2+3y^3$, $x^4+4xy^3+3y^4$, $x^4-2x^3y+2x^2y^2+2xy^3-3y^4$.

83. $x^4-x^3-3x^2+5x-2$, $x^4+3x^3-3x^2-7x+6$, $x^4+x^3-3x^2-x+2$.

84. $x^3-2ax^2-2a^2x-3a^3$, $2x^3-ax^2-a^2x-3a^3$, $x^5+a^2x^3-a^3x^2-a^5$.

85. $2x^5-6x^4+8x^2$, $6x^5+12x^4-30x^3-36x^2$, $4x^6-8x^5-4x^4+8x^3$.

By addition or subtraction, after multiplying by a numerical factor if necessary, find the common factor of the first degree of

86. x^2-5x+6, $x^2+4x-21$.

87. x^2-8x-9, $x^2+3x-108$.

88. $2x^2+5x-3$, $6x^2+x-2$.

89. $4x^2+9x+2$, $8x^2-10x-3$.

90. $2x^3-9x-27$, x^3+x-30.

91. x^3+x^2-12, x^3-x^2-4.

92. x^3+x^2-2, x^3+2x^2-3.

93. x^3+x^2+x-14, x^3-x^2-x-2.

By addition or subtraction, after multiplying by a numerical factor if necessary, shew that the following expressions have no common factor in x:

94. $4x^3-x^2+x-3$, $2x^3+x^2-x-3$.

95. $3x^4+4x^2+4$, $3x^4+2x^2+2$.

96. $5x^4+2x^3+2x^2+2x+5$, $2x^4-5x^3-5x^2-5x+2$.

Ascertain what numerical values a can have so that the following expressions may contain a common factor:

97. ax^2-5x+3, $2ax^2-8x+3$.

98. ax^2-x-2, $7ax^2+11x-2$.

99. x^2-5x+a, $x^2-7x+2a$.

100. x^2+6x+a, $x^2+12x+3a$.

CHAPTER VIII.

LOWEST COMMON MULTIPLE.

58. A common multiple of two or more algebraical expressions is an expression exactly divisible by each of them, and their **Lowest Common Multiple** is the expression of lowest dimensions which is exactly divisible by each of them. Every common multiple of any number of algebraical expressions is therefore a multiple of their lowest common multiple. The lowest common multiple is often written in the abbreviated form L.C.M.

The L.C.M. of simple expressions is obvious from the definition and can be written down by inspection. It consists of *the least common multiple of the numerical coefficients prefixed to the product of the lowest power of each letter which is exactly divisible by every power of that letter occurring in any of the given expressions.*

Thus the L.C.M. of a^4b^2, a^3b^5 is a^4b^5;

of $4a^2bc^3$, $7a^4b^3c$, $2ab^2c^4$ is $28a^4b^3c^4$;

of $9x^3y^3z^5$, $3x^2yz^4$, $27x^2y^2z^2$, $81x^4y^3z$ is $81x^4y^3z^5$.

59. The L.C.M. of compound expressions which by inspection admit of resolution into factors can also at once be written down in accordance with the definition. It consists of *the least common multiple of the numerical coefficients prefixed to the product of the lowest power of each factor which is exactly divisible by every power of that factor occurring in any of the given expressions.*

Thus the L.C.M. of a^2-b^2, a^3+b^3,

i.e. of $(a+b)(a-b)$, $(a+b)(a^2-ab+b^2)$,

is $(a+b)(a-b)(a^2-ab+b^2)$;

of $9a^3b-4ab^3$, $9a^3b-12a^2b^2+4ab^3$, $27a^4-8ab^3$,

i.e. of $ab(9a^2-4b^2)$, $ab(9a^2-12ab+4b^2)$, $a(27a^3-8b^3)$,

i.e. of $ab(3a-2b)(3a+2b)$, $ab(3a-2b)^2$, $a(3a-2b)(9a^2+6ab+4b^2)$,
is $ab(3a-2b)^2(3a+2b)(9a^2+6ab+4b^2)$;
of $3x^3-3x^2-6x$, $6x^3+6x^2-36x$, $2x^4-8x^3+8x^2$,
i.e. of $3x(x^2-x-2)$, $6x(x^2+x-6)$, $2x^2(x^2-4x+4)$,
i.e. of $3x(x+1)(x-2)$, $6x(x-2)(x+3)$, $2x^2(x-2)^2$,
is $6x^2(x+1)(x-2)^2(x+3)$.

60. The only case remaining for consideration is when the given compound expressions cannot readily be resolved into their component factors. In this case we can determine the H.C.F. by the method of last chapter, and dividing each of the expressions by this H.C.F. we obtain the other factor of each expression; the resolution into factors being now accomplished so far as is required for our purpose, we may write down the L.C.M. as before by inspection.

Example. Find the L.C.M. of

$$x^3+2x^2-5x-6 \text{ and } x^3-x^2-5x-3.$$

By the method of last chapter the H.C.F. of the two expressions is found to be $x+1$; dividing the first expression by $x+1$ we get x^2+x-6, and dividing the second by $x+1$ we get x^2-2x-3, so that

$$x^3+2x^2-5x-6=(x+1)(x^2+x-6)$$

and $$x^3-x^2-5x-3=(x+1)(x^2-2x-3).$$

Hence the L.C.M. is $(x+1)(x^2+x-6)(x^2-2x-3)$.

In an example like this where after division by the H.C.F. the other factor thus determined is a quadratic expression, this factor again may frequently be resolved into its component factors and the final result may be presented in a simpler form.

In the present example

$$x^2+x-6=(x-2)(x+3) \text{ and } x^2-2x-3=(x+1)(x-3),$$

so that the L.C.M. may be written in the neater form

$$(x+1)^2(x-2)(x-3)(x+3).$$

61. We may however obtain a general rule for the treatment of such expressions which is equivalent to the directions given in the last Article. Let A, B denote two algebraical expressions and let C be their H.C.F., so that $A=aC$, $B=bC$. Then from the nature of the H.C.F. a and b contain no common factor; hence the L.C.M. of aC and bC must be the product abC, which is therefore the L.C.M. of A and B required. Also

$$abC=\frac{abC^2}{C}=\frac{aC\times bC}{C}=\frac{A\times B}{C},$$

so that we are led to the following rule:

The L. C. M. *of any two algebraical expressions is the quotient obtained by dividing their product by their* H. C. F.; or, *is the product of one expression and of the quotient obtained by dividing the other expression by the* H. C. F.

Example. Find the L. C. M. of

$$x^3-x^2-16x-20 \text{ and } x^3+x^2-4x-4.$$

By the ordinary rule we find the H. C. F. of these two expressions to be $x+2$; dividing $x^3-x^2-16x-20$ by $x+2$ we get $x^2-3x-10$, and multiplying x^3+x^2-4x-4 by $x^2-3x-10$ we get

$$x^5-2x^4-17x^3-2x^2+52x+40,$$

which is the L. C. M. required.

In this example as before the complete resolution of the expressions may be effected and so the L. C. M. may be put in a simpler form. For having discovered the H. C. F. to be $x+2$, we find the other factors of the expressions to be $x^2-3x-10$, i.e. $(x+2)(x-5)$, and x^2-x-2, i.e. $(x+1)(x-2)$, respectively, so that

$$x^3-x^2-16x-20=(x+2)^2(x-5),$$

$$x^3+x^2-4x-4=(x+1)(x-2)(x+2);$$

hence the L. C. M. is $(x+1)(x-2)(x+2)^2(x-5)$, which by multiplying out produces $x^5-2x^4-17x^3-2x^2+52x+40$.

62. In cases where the L. C. M. of more than two algebraical expressions is required, and they cannot immediately be resolved into their component factors by inspection, we must extend the general method explained in Art. 60, 61.

From the nature of the L. C. M. it is plain that every common multiple of three or more expressions must be a multiple of the L. C. M. of any two of the expressions. Hence, to obtain the L. C. M. of three or more expressions, *we must first find the* L. C. M. *of any two of the expressions, then the* L. C. M. *of this* L. C. M. *and of a third expression, and so on; the last* L. C. M. *so found is the* L. C. M. *of all the expressions.*

Example. Find the L. C. M. of

$$x^3-x^2-8x+12,\ x^3+4x^2-3x-18,\ \text{and}\ x^4+6x^3-54x-81.$$

The L. C. M. of the first and second expressions is found by the rule to be $x^4+2x^3-11x^2-12x+36$, and the L. C. M. of this expression and of the third expression is by the rule

$$x^6+2x^5-20x^4-30x^3+135x^2+108x-324,$$

which is therefore the L. C. M. required.

It is however very seldom that the adoption of this cumbrous rule is necessary, and in the present example it may be avoided. For in the process of finding the L.C.M. of $x^3-x^2-8x+12$ and $x^3+4x^2-3x-18$ we find that the H.C.F. of these two expressions is x^2+x-6, i.e. $(x-2)(x+3)$, so that the other factors are respectively $x-2$, $x+3$, and we have

$$x^3-x^2-8x+12=(x-2)^2(x+3),$$

$$x^3+4x^2-3x-18=(x-2)(x+3)^2.$$

Or if we begin with finding the L.C.M. of the second and third expressions we find that the H.C.F. of these expressions is x^2+6x+9, i.e. $(x+3)^2$, so that the other factors are respectively $x-2$ and x^2-9, i.e. $(x+3)(x-3)$, and we have

$$x^4+6x^3-54x-81=(x+3)^3(x-3).$$

Hence by inspection the L.C.M. required is

$$(x-2)^2(x-3)(x+3)^3,$$

which by multiplication produces

$$x^6+2x^5-20x^4-30x^3+135x^2+108x-324.$$

63. It should be borne in mind that the rules for finding the H.C.F. and L.C.M. are only intended to be applied when the expressions under treatment cannot be resolved into their component factors by inspection, and that a large majority of examples in the subject can easily be worked, and should be worked, by resolution into factors.

EXAMPLES.

Write down the L.C.M. of

1. a^3b^6, a^4b^2.
2. $4a^3bc^2$, $12a^2b^4$.
3. $42x^2y$, $14xy^3$.
4. $6x^2y^4$, $30x^3y^3$.
5. $3a^2b^3c$, $9a^3bc^4$, $6a^4b^2c^5$.
6. $25a^4bc$, $125ab^4c^3$, $100b^3c^4$.
7. $34a^2x^4y$, $85a^4xy^3$, $17ax^3y^4$.
8. x^3yz^2, $2x^2y^4$, $4x^2y^2z^2$, $6xyz$.
9. $24x^4y^3z^2$, $36xy^5z^3$, $72x^2y^2z^5$, $9x^5yz^4$.
10. $27xy^2z^4$, $3x^3yz^3$, $9x^4y^4z$, $18x^6$.

By resolution into factors find the L.C.M. of

11. a^2-b^2, a^3+b^3.
12. $3(a-b)^2$, $4(a^3-b^3)$.
13. a^3-b^3, $(a+b)(a^2+ab+b^2)$.
14. $2a^4-2a^3+2a^2$, $6a^4+6a$.
15. a^3+b^3, $a^3-a^2b+ab^2$.

16. a^2-b^2, a^3+b^3, a^3-b^3.

17. a^3+b^3, a^3-b^3, $a^4+a^2b^2+b^4$.

18. $a^3b^3+2a^2b^2+ab$, $a^4b^4+3a^3b^3+3a^2b^2+ab$.

19. $a(b-c)+b(c-a)$, $b(b-c)+a(c-a)$.

20. $a^2(b-c)+b^2(c-a)$, $bc(b-c)+ca(c-a)$.

21. $ac-bd-bc+ad$, $ac+bd-bc-ad$.

22. $a^2+b^2-c^2+2ab$, $a^2-b^2-c^2-2bc$.

23. $a^2+b^2+2bc+2ca+2ab$, $a^2-b^2-2bc+2ca$.

24. $a^3-3a^2b+3ab^2-b^3$, $a^3-2a^2b+ab^2$.

25. $12a^5b-12ab^5$, $15a^6+15b^6$.

26. x^2-3x+2, x^2+x-6.

27. x^2+4x-5, $x^2-4x-45$.

28. x^2-6x+9, $x^2+5x-24$.

29. $x^2-2xy-8y^2$, $x^2-xy-12y^2$.

30. $x^2+4ax-77a^2$, $x^2-4ax-165a^2$.

31. $x^2+9ax-36a^2$, $x^2+23ax+132a^2$.

32. $3x^3-15x^2+18x$, $2x^3+14x^2-36x$.

33. $12x^2+2x-4$, $6x^2-8x-8$.

34. $12x^2-x-63$, $4x^2+x-18$.

35. $2x^2+15x-8$, $2x^2+23x-12$.

36. $12x^2-x-6$, $8x^2-2x-3$.

37. $6x^2+xy-40y^2$, $9x^2-64y^2$.

38. $9x^2+3xy-12y^2$, $6x^2-xy-12y^2$.

39. $12x^2-11ax-5a^2$, $8x^2-6ax-5a^2$.

40. $11x^2-ax-12a^2$, $11x^2-23ax+12a^2$.

41. $12a^2+40a-7$, $10a^2+29a-21$.

42. $9a^2+5ab-26b^2$, $10a^2+9ab-22b^2$.

43. $abx^2-(a^2-b^2)x-ab$, $abx^2+(a^2+b^2)x+ab$.

44. $ax^2+(a^2-b^2)x-ab^2$, $ax^2-(a^2+b^2)x+ab^2$.

45. $12a^2x^3+33ax^2-9x$, $12a^2x^3+32ax^2-12x$.

46. $a^4-a^3b+ab^3-b^4$, $a^4+a^3b-ab^3-b^4$, $a^4+a^2b^2+b^4$.

47. x^2+3x-4, x^2+5x+4, x^2-16.

48. $x^2+14x+49$, $x^2+5x-14$, $x^2+4x-21$.

49. $x^2+4xy-12y^2$, $x^2+2xy-24y^2$, $x^2-2xy-48y^2$.

50. $2x^2+11x-21$, $6x^2-7x-3$, $6x^2-11x+3$.

51. $5x^3+3x^2-2x$, $10x^3+11x^2-6x$, $5x^3-7x^2+2x$.

52. x^2-25a^2, $x^2+5ax+25a^2$, x^3-125a^3.

53. x^4-xy^3, x^3y-y^4, $x^4+x^2y^2+y^4$.

54. x^2-4x+3, x^3-4x+3, x^3-4x^2+3, x^4-4x^2+3.

55. $2(x^3y^2-x^2y^3)$, $4(x^4y^2-x^2y^4)$, $8(x^5y^2-x^2y^5)$, $12(x^6y^2-x^2y^6)$.

56. $x^2-(b+c)x+bc$, $x^2-(c+a)x+ca$, $x^2-(a+b)x+ab$.

57. $x^2-(b-c)x-bc$, $x^2+(c-a)x-ca$, $x^2-(a+b)x+ab$.

58. $x^2-(b+c)x+bc$, $x^2-(c-a)x-ca$, $x^2+(a-b)x-ab$,
$x^3+(a-b-c)x^2+(bc-ca-ab)x+abc$.

59. $(x+y+z)(x+y-z)(x-y+z)$, $(y+z-x)(x^2-y^2-z^2+2yz)$,
$4y^2z^2-(x^2-y^2-z^2)^2$.

60. x^2-y^2, x^2+y^2, x^4-y^4, x^4+y^4, x^8-y^8.

Find the L.C.M. of

61. $a^3+3a^2b-ab^2-3b^3$, $a^3-a^2b-9ab^2+9b^3$.

62. $a^3+2a^2b-7ab^2+4b^3$, $a^3+a^2b-10ab^2+8b^3$.

63. a^3-7a+6, $a^4+4a^3-7a^2-22a+24$.

64. $a^3+a^2b-ab^2+b^3$, $a^4-a^3b+ab^3-b^4$.

65. x^3+x^2-5x+3, x^3-2x^2-5x+6.

66. x^3-2x^2-7x-4, $x^3-x^2-10x-8$.

67. $x^3-2a^2x+4a^3$, $x^3+3ax^2-8a^2x+10a^3$.

68. $x^3-xy^2-6y^3$, $x^3-2x^2y-5xy^2-12y^3$.

69. $6x^3-x^2-10x-3$, $12x^3+7x^2-8x-3$.

70. $3x^3+x^2+x-5$, $6x^3-x^2-2x-15$.

71. $7x^3-16x^2y-34xy^2+24y^3$, $9x^3-25x^2y-26xy^2+22y^3$.

72. x^4-6x^2+8x-3, $x^4+2x^3-4x^2-2x+3$.

73. $x^4-2x^3-12x^2-7x+2$, $x^4-3x^3-6x^2-13x+3$.

74. $x^4+2x^3y-5x^2y^2+2y^4$, $x^4+2x^3y-4x^2y^2-2xy^3+3y^4$.

75. $6x^4+x^3+6x^2+3x+2$, $6x^4+x^3+2x^2+5x-2$.

76. $9x^6-30ax^5+12a^2x^4+24a^3x^3$, $12x^6+20ax^5-64a^2x^4-48a^3x^3$.

77. $8x^4y+16x^3y^2-22x^2y^3+6xy^4$, $12x^4y+36x^3y^2-3x^2y^3-9xy^4$.

78. x^3+x^2+x-3, x^3-x-6, x^3-x^2-3x-9.

79. $x^3+8ax^2+20a^2x+16a^3$, $x^3-12a^2x-16a^3$, $x^3+2ax^2-4a^2x-8a^3$.

80. $x^3-2x^2y-xy^2-6y^3$, $x^3-3x^2y-2xy^2-8y^3$, $x^3-4x^2y-3xy^2-10y^3$,
$x^4-5x^3y+5x^2y^2-3xy^3+18y^4$.

CHAPTER IX.

FRACTIONS.

64. It has been explained in Art. 4 that the operation of division is indicated either by the symbol $\div$ or by a straight line drawn between two quantities of which the upper is the dividend and the lower the divisor. Thus if a when divided by b produces the quotient c, the operation is expressed by the equation $a \div b = c$ or $\frac{a}{b} = c$; and in this equation c is such a quantity that when multiplied by b it produces a. Hence we may define the meaning of the algebraical fraction $\frac{a}{b}$ to be that *it is such a quantity as when multiplied by b produces a.* This definition is also in accordance with our arithmetical conception of fractions; for in arithmetic a quantity such as $\frac{3}{4}$ is that quantity which when multiplied by 4 produces 3, and similarly in any other instance.

In arithmetic we more usually define a fraction such as $\frac{3}{4}$ to mean that the unit is divided into 4 equal parts and that 3 of these parts are taken; and so we might anticipate that in algebra the fraction $\frac{a}{b}$ would mean that the unit is divided into b equal parts and that a of these parts are taken, and this is intelligible so long as we restrict a and b to be positive integers. But if a or b or both be fractional or negative the definition becomes unintelligible, and we are driven to adopt such a definition as will include fractions in algebra as well as in arithmetic, and all possible values of numerator and denominator, positive or negative, integral or fractional. Such a definition as we seek is provided by the statement that the fraction $\frac{a}{b}$ is such a quantity

as when multiplied by b produces a; this definition includes all possible values of a and b, positive or negative, integral or fractional.

65. We shall now shew that *the value of an algebraical fraction is not changed by multiplying both numerator and denominator by any the same quantity.*

If c be a quantity expressed in the fractional form $\frac{a}{b}$, then by the definition of a fraction (Art. 64) we know that c or $\frac{a}{b}$ is such a quantity as when multiplied by b produces a, so that

$$c \times b = \frac{a}{b} \times b = a;$$

hence
$$c \times b \times x = a \times x,$$

and
$$\frac{c \times b \times x}{b \times x} = \frac{a \times x}{b \times x};$$

$$\therefore\ c = \frac{a \times x}{b \times x},$$

$$\text{i.e. } \frac{a}{b} = \frac{ax}{bx}.$$

Hence also it follows conversely that *the value of a fraction is not changed if we divide both numerator and denominator by any the same quantity.*

66. As the value of a fraction is thus not changed by multiplying numerator and denominator by the same quantity, it follows that by employing suitable factors we may make any number of fractions have the same denominator. For such fractions are equivalent to a new set of fractions of which the common denominator is the product of all the denominators, and each new numerator is formed by multiplying the original numerator by all the denominators except its own. This transformation will not however necessarily or generally give the lowest common denominator; in order that the fractions may be reduced so as to have the lowest common denominator, the L.C.M. of the denominators must be found and be made the new common denominator, and each new numerator will then be the product of the original numerator and of the quotient obtained by dividing this L.C.M. by the original denominator.

Thus the fractions $\frac{a}{b}, \frac{c}{d}$ are equivalent to $\frac{ad}{bd}, \frac{bc}{bd}$, fractions having a common denominator and also the lowest common denominator since the product of the denominators is also their L. C. M.;

$$\frac{x}{(x+y)^2}, \quad \frac{y}{x^2-y^2}$$

are equivalent to $\frac{x(x^2-y^2)}{(x+y)^2(x^2-y^2)}, \frac{y(x+y)^2}{(x+y)^2(x^2-y^2)}$,

fractions having a common denominator but not the lowest common denominator, and they are also equivalent to

$$\frac{x(x-y)}{(x+y)^2(x-y)}, \quad \frac{y(x+y)}{(x+y)^2(x-y)},$$

fractions having the lowest common denominator, which is the L. C. M. of the original denominators. Similarly

$$\frac{1}{x^2-3x+2}, \quad \frac{1}{x^2+3x-4}, \quad \frac{1}{x^2+2x-8}$$

are equivalent to

$$\frac{(x^2+3x-4)(x^2+2x-8)}{(x^2-3x+2)(x^2+3x-4)(x^2+2x-8)},$$

$$\frac{(x^2+2x-8)(x^2-3x+2)}{(x^2-3x+2)(x^2+3x-4)(x^2+2x-8)},$$

$$\frac{(x^2-3x+2)(x^2+3x-4)}{(x^2-3x+2)(x^2+3x-4)(x^2+2x-8)},$$

but in this case again the new fractions have not the lowest common denominator; writing the original fractions in the form

$$\frac{1}{(x-1)(x-2)}, \quad \frac{1}{(x+4)(x-1)}, \quad \frac{1}{(x-2)(x+4)},$$

the L. C. M. of the denominators is $(x-1)(x-2)(x+4)$ by inspection, and the given fractions are equivalent to the fractions

$$\frac{x+4}{(x-1)(x-2)(x+4)}, \quad \frac{x-2}{(x-1)(x-2)(x+4)}, \quad \frac{x-1}{(x-1)(x-2)(x+4)},$$

with the lowest common denominator.

67. Also as the value of a fraction is not changed by dividing numerator and denominator by the same quantity, we may simplify any fraction by removing common factors from the numerator and denominator. A fraction is said to be in its **lowest terms** when there are no common factors in the numerator and denominator; hence to reduce a fraction to its

lowest terms we must divide the numerator and denominator by their H.C.F.

Thus by removing the H.C.F. from the numerator and denominator the fraction

$$\frac{2a^3b^2c^4}{8a^3bc^5} \text{ is equivalent to } \frac{b}{4c};$$

$$\frac{(a+b)^2}{a^2-b^2} \text{ is equivalent to } \frac{a+b}{a-b};$$

$$\frac{27x^3+125y^3}{9x^2-25y^2}, \text{ i.e. } \frac{(3x+5y)(9x^2-15xy+25y^2)}{(3x+5y)(3x-5y)},$$

is equivalent to $$\frac{9x^2-15xy+25y^2}{3x-5y};$$

$$\frac{2x^4+5x^3-11x^2-31x-10}{4x^4+17x^3+13x^2-13x-6}, \text{ i.e. } \frac{(x^3+5x^2+7x+2)(2x-5)}{(x^3+5x^2+7x+2)(4x-3)},$$

is equivalent to $$\frac{2x-5}{4x-3};$$

all the fractions thus obtained being in their lowest terms.

68. We are now in a position to add or subtract fractions, and the rule is:

Find the L.C.M. *of the denominators; reduce the fractions so as to have this* L.C.M. *for their common denominator and add or subtract the numerators.*

Example 1.

$$\frac{ab}{a^3+b^3}+\frac{1}{a+b}=\frac{ab}{a^3+b^3}+\frac{a^2-ab+b^2}{a^3+b^3}=\frac{ab+a^2-ab+b^2}{a^3+b^3}=\frac{a^2+b^2}{a^3+b^3}.$$

Example 2.

$$\frac{x^2+y^2}{x^4+x^2y^2+y^4}-\frac{1}{x^2+xy+y^2}=\frac{x^2+y^2}{x^4+x^2y^2+y^4}-\frac{x^2-xy+y^2}{x^4+x^2y^2+y^4}$$
$$=\frac{x^2+y^2-x^2+xy-y^2}{x^4+x^2y^2+y^4}=\frac{xy}{x^4+x^2y^2+y^4}.$$

Example 3.

$$\frac{1}{x^2-4x+4}+\frac{1}{x^2+x-6}-\frac{1}{x^2-5x+6}$$
$$=\frac{1}{(x-2)^2}+\frac{1}{(x-2)(x+3)}-\frac{1}{(x-2)(x-3)}$$
$$=\frac{(x-3)(x+3)+(x-2)(x-3)-(x-2)(x+3)}{(x-2)^2(x-3)(x+3)}$$
$$=\frac{x^2-9+x^2-5x+6-x^2-x+6}{(x-2)^2(x-3)(x+3)}=\frac{x^2-6x+3}{(x-2)^2(x-3)(x+3)}.$$

Example 4. $\dfrac{1}{x-a}-\dfrac{1}{x+a}-\dfrac{2a}{x^2+a^2}-\dfrac{4a^3}{x^4+a^4}$

$$=\frac{x+a-(x-a)}{x^2-a^2}-\frac{2a}{x^2+a^2}-\frac{4a^3}{x^4+a^4}$$

$$=\frac{2a}{x^2-a^2}-\frac{2a}{x^2+a^2}-\frac{4a^3}{x^4+a^4}$$

$$=\frac{2a(x^2+a^2)-2a(x^2-a^2)}{x^4-a^4}-\frac{4a^3}{x^4+a^4}=\frac{4a^3}{x^4-a^4}-\frac{4a^3}{x^4+a^4}$$

$$=\frac{4a^3(x^4+a^4)-4a^3(x^4-a^4)}{x^8-a^8}=\frac{8a^7}{x^8-a^8}.$$

As in the last example it is sometimes more convenient not to reduce all the fractions to a common denominator at once but to go through the process gradually.

69. It is obvious that the value of a fraction remains unaltered if we change the sign of both numerator and denominator, for this is equivalent to multiplying numerator and denominator by the same quantity -1.

Example.

$$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}$$

$$=-\frac{1}{a(a-b)(c-a)}-\frac{1}{b(b-c)(a-b)}-\frac{1}{c(c-a)(b-c)}$$

$$=-\frac{bc(b-c)+ca(c-a)+ab(a-b)}{abc(b-c)(c-a)(a-b)}$$

$$=\frac{(b-c)(c-a)(a-b)}{abc(b-c)(c-a)(a-b)}$$ by Chap. VI. Ex. 41, $=\dfrac{1}{abc}$.

In this example

$$\frac{1}{a(a-b)(a-c)}=\frac{-1}{-a(a-b)(a-c)}$$

$$=\frac{-1}{a(a-b)\times-(a-c)}=\frac{-1}{a(a-b)(c-a)}=-\frac{1}{a(a-b)(c-a)},$$

and similarly with the rest.

70. We must next investigate the product of two algebraical fractions $\dfrac{a}{b}$, $\dfrac{c}{d}$; let the required product be x, so that

$$x=\frac{a}{b}\times\frac{c}{d}.$$

Multiplying by the product $b \times d$ and remembering that the order of factors in a product is a matter of indifference, we have

$$x \times b \times d = \left(\frac{a}{b} \times b\right) \times \left(\frac{c}{d} \times d\right) = a \times c$$

by the definition of a fraction; dividing by the product $b \times d$ we have

$$x = \frac{a \times c}{b \times d} = \frac{ac}{bd}.$$

Hence *the product of two algebraical fractions is a fraction whose numerator is the product of the numerators and denominator the product of the denominators.*

The rule thus proved for the product of two algebraical fractions may easily be extended to three or more fractions. For by this rule

$$\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ac}{bd} \times \frac{e}{f} = \frac{ace}{bdf}$$

by the same rule; and a similar result follows for the product of any number of fractions. Hence *the product of any number of fractions is a fraction whose numerator is the product of the numerators and denominator the product of the denominators.*

From this rule it follows at once that $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$ and $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$ and so on; and generally that $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, where n is any positive integer.

Example 1. Multiply

$$\frac{3a^2}{4bc} \text{ by } \frac{8b^2c^2}{9a^4}.$$

$$\frac{3a^2}{4bc} \times \frac{8b^2c^2}{9a^4} = \frac{3a^2 \times 8b^2c^2}{4bc \times 9a^4} = \frac{12a^2bc \times 2bc}{12a^2bc \times 3a^2} = \frac{2bc}{3a^2}.$$

Example 2. Multiply $\frac{(x+y)^3}{x^3-y^3}$ by $\frac{x^2+xy+y^2}{x^2-y^2}$.

$$\frac{(x+y)^3}{x^3-y^3} \times \frac{x^2+xy+y^2}{x^2-y^2} = \frac{(x+y)^3 \times (x^2+xy+y^2)}{(x^3-y^3) \times (x^2-y^2)}$$

$$= \frac{(x+y)\,(x^2+xy+y^2) \times (x+y)^2}{(x+y)\,(x^2+xy+y^2) \times (x-y)^2} = \frac{(x+y)^2}{(x-y)^2} = \left(\frac{x+y}{x-y}\right)^2.$$

Example 3. Multiply together

$$\frac{x^2-(a+b)x+ab}{x^2-c^2}, \quad \frac{x^2+(b+c)x+bc}{x^2-a^2}, \quad \frac{x^2-(c-a)x-ca}{x^2-b^2}.$$

$$\frac{x^2-(a+b)x+ab}{x^2-c^2} \times \frac{x^2+(b+c)x+bc}{x^2-a^2} \times \frac{x^2-(c-a)x-ca}{x^2-b^2}$$

$$= \frac{(x-a)(x-b)}{(x-c)(x+c)} \times \frac{(x+b)(x+c)}{(x-a)(x+a)} \times \frac{(x-c)(x+a)}{(x-b)(x+b)}$$

$$= \frac{(x-a)(x-b)\times(x+b)(x+c)\times(x-c)(x+a)}{(x-c)(x+c)\times(x-a)(x+a)\times(x-b)(x+b)}$$

$$= \frac{(x-a)(x+a)(x-b)(x+b)(x-c)(x+c)}{(x-a)(x+a)(x-b)(x+b)(x-c)(x+c)} = 1.$$

71. If we have to divide a fraction $\frac{a}{b}$ by another fraction $\frac{c}{d}$, let x be the quotient required; then by the nature of division $\frac{a}{b}$ is the product of x and $\frac{c}{d}$, so that $x \times \frac{c}{d} = \frac{a}{b}$.

Multiplying by $\frac{d}{c}$, we have $x \times \frac{c}{d} \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c}$;

i.e. $$x \times \frac{cd}{dc} = \frac{ad}{bc};$$

$$\therefore \quad x = \frac{ad}{bc}.$$

Thus *to divide a fraction by another fraction invert the divisor and then multiply.*

A fraction when inverted is said to be the *reciprocal* of what it was before, so that $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$ and $\frac{1}{a}$ is the reciprocal of a or $\frac{a}{1}$.

Example 1. Divide $\frac{3a^2}{5b^2}$ by $\frac{9a}{10b}$.

$$\frac{3a^2}{5b^2} \div \frac{9a}{10b} = \frac{3a^2}{5b^2} \times \frac{10b}{9a} = \frac{3a^2 \times 10b}{5b^2 \times 9a} = \frac{15ab \times 2a}{15ab \times 3b} = \frac{2a}{3b}.$$

Example 2. Divide $\frac{x^2-5x+4}{2x^2-x-3}$ by $\frac{x^2-x-12}{3x^2+x-2}$.

$$\frac{x^2-5x+4}{2x^2-x-3} \div \frac{x^2-x-12}{3x^2+x-2} = \frac{x^2-5x+4}{2x^2-x-3} \times \frac{3x^2+x-2}{x^2-x-12}$$

$$= \frac{(x^2-5x+4)\times(3x^2+x-2)}{(2x^2-x-3)\times(x^2-x-12)} = \frac{(x-1)(x-4)\times(x+1)(3x-2)}{(x+1)(2x-3)\times(x+3)(x-4)}$$

$$= \frac{(x+1)(x-4)\times(x-1)(3x-2)}{(x+1)(x-4)\times(x+3)(2x-3)} = \frac{(x-1)(3x-2)}{(x+3)(2x-3)}.$$

72. More difficult examples in fractions are when numerator or denominator consists of fractions, or when numerator or denominator is partly integral in form and partly fractional; fractions of this kind are called *complex fractions*, and the simplification, when required, should be performed gradually.

Example 1. Simplify $\dfrac{\frac{a^3}{x^3}-\frac{x^3}{a^3}}{\frac{a^2}{x^2}+1+\frac{x^2}{a^2}}$.

By the formula for the difference of two cubes the numerator

$$= \left(\frac{a}{x}-\frac{x}{a}\right)\left(\frac{a^2}{x^2}+1+\frac{x^2}{a^2}\right);$$

hence the value of the fraction is $\frac{a}{x}-\frac{x}{a}$.

Example 2. Simplify $\dfrac{x-4+\frac{3}{x}}{x+2-\frac{15}{x}}$.

Multiplying numerator and denominator by x the fraction takes the form $\frac{x^2-4x+3}{x^2+2x-15}$, and this is equivalent to $\frac{(x-3)(x-1)}{(x-3)(x+5)}$, i.e. to $\frac{x-1}{x+5}$.

Example 3. Simplify $\dfrac{\frac{a}{a-x}-\frac{x}{a+x}}{\frac{a}{a+x}+\frac{x}{a-x}}$.

The numerator $= \frac{a(a+x)-x(a-x)}{a^2-x^2} = \frac{a^2+x^2}{a^2-x^2}$;

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The denominator

$$= \frac{a(a-x)+x(a+x)}{a^2-x^2} = \frac{a^2+x^2}{a^2-x^2};$$

thus the numerator is equal to the denominator and therefore the value of the fraction is 1.

Example 4. Simplify $\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{1+\cfrac{2x}{1-x}}}}$.

The given fraction

$$= \cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{\cfrac{1+x}{1-x}}}} = \cfrac{1}{1-\cfrac{1}{1-\cfrac{1-x}{1+x}}} = \cfrac{1}{1-\cfrac{1}{\cfrac{2x}{1+x}}} = \cfrac{1}{1-\cfrac{1+x}{2x}} = \cfrac{1}{\cfrac{x-1}{2x}} = \frac{2x}{x-1}.$$

73. The solution of problems in fractions is greatly facilitated by a readiness in resolving algebraical expressions into their component factors in accordance with laws and formulæ that have been explained in former chapters. As a general rule the factors of the numerator and denominator of a fraction become apparent on inspection, and in this case the H.C.F. of the numerator and denominator is known and the fraction can at once be written down in its lowest terms. And when several fractions are under treatment the component factors of their denominators are generally apparent, so that the L.C.M. of the denominators becomes known by inspection, and the fractions are reduced to this L.C.M. as a common denominator by multiplying each numerator by the quotient obtained from the division of the L.C.M. by the denominator of the fraction. When the factors of the numerator and denominator of a fraction are not obvious, their H.C.F. must be found by the rule; and the discovery of this H.C.F. will frequently lead to the discovery of the other component factors, if any, and will always do so when the remaining expressions are quadratic in form.

Similarly when the factors of the denominators of any number of fractions are not obvious on inspection, the L.C.M. of the denominators must be found by the rule; it is however in very rare instances that the employment of the rule is necessary, and its use should always be avoided when possible.

Results are said to be obtained *by inspection* when they are obtained by transformation or the use of formulæ and without the mechanical application of rules.

EXAMPLES.

By inspection reduce to their lowest terms :

1. $\frac{4a^2b^3c^6}{32ab^4c^8}$.
2. $\frac{7x^3y}{42xy^3}$.
3. $\frac{17x^4y^2z^3}{51x^6yz^2}$.
4. $\frac{9a^3bc^4x^2y^2z^5}{117a^2bc^3x^3y^4z^7}$.
5. $\frac{a^2-b^2}{(a+b)^2}$.
6. $\frac{a^2+ab+b^2}{a^3-b^3}$.
7. $\frac{a^2+2ab+b^2}{a^3+b^3}$.
8. $\frac{a^4-a}{a^4+a^2+1}$.
9. $\frac{a^2+b^2-c^2+2ab}{a^2-b^2+c^2+2ca}$.
10. $\frac{a^2-b^2-ac+bc}{a^2-b^2-c^2+2bc}$.
11. $\frac{a^2+b^2-2bc-2ca+2ab}{a^2+b^2-4c^2+2ab}$.
12. $\frac{ac+bd+bc+ad}{ac-bd-bc+ad}$.
13. $\frac{a^3+a^2b+ab^2+b^3}{a^3-a^2b-ab^2+b^3}$.
14. $\frac{a^3b+b^4}{a^4+a^2b^2+b^4}$.
15. $\frac{a^3b-ab^3}{a^4-2a^2b^2+b^4}$.
16. $\frac{x^2-x-42}{x^2+x-30}$.
17. $\frac{x^2-7x+6}{x^2-11x+10}$.
18. $\frac{x^2-16}{x^2+8x+16}$.
19. $\frac{5x^2+25x}{25-x^2}$.
20. $\frac{x^2+(a+b)x+ab}{x^2-(c-a)x-ca}$.
21. $\frac{x^2-(a-b)x-ab}{x^2+(b-c)x-bc}$.
22. $\frac{x^2-a^2}{x^2+(a-b)x-ab}$.
23. $\frac{x^2-4ax+3a^2}{x^2+4ax-21a^2}$.
24. $\frac{x^2+5xy-36y^2}{x^2-8xy+16y^2}$.
25. $\frac{a^2x^2-1}{2ax^2-(a^2-2)x-a}$.
26. $\frac{2x^2+x-15}{2x^2-11x+15}$.
27. $\frac{3x^2-x-4}{4x^2+x-3}$.
28. $\frac{6x^2+5x-6}{10x^2+9x-9}$.
29. $\frac{8x^2-26xy-7y^2}{6x^2-19xy-7y^2}$.
30. $\frac{ab(x^2-1)+(a^2-b^2)x}{a^2x^2-b^2}$.
31. $\frac{abx^2-(a^2b^2+1)x+ab}{cx^2-(abc+1)x+ab}$.
32. $\frac{x^3+2x^2-4x-8}{x^3-8}$.
33. $\frac{x^3-4x+3}{x^3-1}$.
34. $\frac{x^3-3x^2y+3xy^2-y^3}{3x^2y-6xy^2+3y^3}$.
35. $\frac{(x+y)^3+(x-y)^3}{x^3+3xy^2}$.
36. $\frac{mx^3-n(x^2y+xy^2)+my^3}{mx^3-(m+n)x^2y+mxy^2}$.

37. $\dfrac{x^6-x^2y^4}{x^6+y^6}$.

38. $\dfrac{(x^3+y^3)(x^2+xy+y^2)}{(x^3-y^3)(x^2-xy+y^2)}$.

39. $\dfrac{x^3+(a+b+c)x^2+(bc+ca+ab)x+abc}{x^3-(a-b-c)x^2+(bc-ca-ab)x-abc}$.

40. $\dfrac{x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2}{(x+y+z)^2(x^2+y^2-z^2-2xy)}$.

41. $\dfrac{\frac{x}{3}-\frac{3}{x}}{\frac{x}{3}-2+\frac{3}{x}}$.

42. $\dfrac{x-5+\frac{4}{x}}{x+5-\frac{6}{x}}$.

43. $\dfrac{x-\frac{1}{x}}{x+2+\frac{1}{x}}$.

44. $\dfrac{x-1+\frac{1}{x}}{x^2+1+\frac{1}{x^2}}$.

45. $\dfrac{x^2+1+\frac{1}{x^2}}{x^3-\frac{1}{x^3}}$.

46. $\dfrac{\frac{x^2}{a^2}-1+\frac{a^2}{x^2}}{\frac{x^3}{a^3}+\frac{a^3}{x^3}}$.

47. $\dfrac{x+\frac{1}{2}-\frac{1}{2x}}{x+\frac{3}{2}-\frac{1}{x}}$.

48. $\dfrac{\frac{x}{y}-\frac{y}{x}-2\frac{z}{x}-\frac{z^2}{xy}}{\frac{x}{y}+2+\frac{y}{x}-\frac{z^2}{xy}}$.

49. $\dfrac{\frac{x^2}{z^2}-\frac{y^2}{z^2}+2\frac{y}{z}+1}{\frac{x^2}{z^2}+\frac{y^2}{z^2}-2\frac{xy}{z^2}-1}$.

50. $\dfrac{\frac{x^2}{y^2}+1+\frac{y^2}{x^2}}{\frac{x^4}{y^2}-\frac{y^4}{x^2}}$.

Reduce to their lowest terms :

51. $\dfrac{a^3-8ab^2-3b^3}{a^4-7a^2b^2+b^4}$.

52. $\dfrac{2a^3-13a^2b-5ab^2+b^3}{a^3-48ab^2+7b^3}$.

53. $\dfrac{a^2-2b^2-3c^2-5bc-2ca-ab}{a^2+2b^2-3c^2+bc-2ca-3ab}$.

54. $\dfrac{x^3+x^2+x-3}{x^3+3x^2+5x+3}$.

55. $\dfrac{x^3-2x^2-5x+6}{x^3+x^2-8x-12}$.

56. $\dfrac{x^3-7a^2x+6a^3}{x^3-4ax^2-9a^2x+36a^3}$.

57. $\dfrac{2x^3-9x^2y+27y^3}{3x^3-16x^2y+15xy^2+18y^3}$.

58. $\dfrac{a(b^2+c^2)-b(c^2-a^2)-c(a^2-b^2)-abc}{a(b^2-c^2)-b(c^2-a^2)+c(a^2+b^2)+abc}$.

59. $\dfrac{a^4-5a^3b+7a^2b^2-7ab^3+2b^4}{a^4-a^3b-3a^2b^2+ab^3+2b^4}$.

60. $\dfrac{x^3-x^2-4x+4}{x^4+2x^3-7x^2-8x+12}$.

61. $\dfrac{x^4-3x^3+2x^2-3x+1}{x^4-3x^3+3x^2-3x+2}$.

62. $\dfrac{x^4+2x^3y-3x^2y^2-4xy^3+4y^4}{x^4-2x^3y-3x^2y^2+8xy^3-4y^4}$.

63. $\dfrac{x^5+2ax^4+4a^2x^3+8a^3x^2+16a^4x+32a^5}{x^6-64a^6}$.

64. $\dfrac{3x^8-2x^7y-2x^4y^4+2x^3y^5-y^8}{3x^8-2x^7y+4x^4y^4-2x^3y^5+y^8}$.

65. $\dfrac{x(x-y)+y(y-z)+z(z-x)}{x^3+y^3+z^3-3xyz}$.

By inspection reduce to their lowest common denominator:

66. $\dfrac{a}{b}, \dfrac{a^3}{b^3}$.

67. $\dfrac{a}{b^2c}, \dfrac{ab}{c^4}$.

68. $\dfrac{2}{3a}, \dfrac{3}{4a^2}, \dfrac{5}{6a^3}$.

69. $\dfrac{1}{a-b}, \dfrac{a}{ab+b^2}, \dfrac{b}{a^2-b^2}$.

70. $\dfrac{a}{bc+c^2}, \dfrac{b}{b^2-c^2}, \dfrac{c}{ab-ca}$.

71. $\dfrac{a}{a^3-b^3}, \dfrac{b}{a^3+b^3}, \dfrac{ab}{a^4+a^2b^2+b^4}$.

72. $\dfrac{a^2}{a^2-b^2-c^2+2bc}, \dfrac{b^2}{a^2+b^2-c^2-2ab}$.

73. $\dfrac{1}{a+b+c}, \dfrac{1}{a-b+c}, \dfrac{c}{a^2-b^2+c^2+2ca}$.

74. $\dfrac{1}{5a+3b}, \dfrac{a^2}{125a^3+27b^3}, \dfrac{a}{25a^2-15ab+9b^2}$.

75. $\dfrac{1}{2a-7b}, \dfrac{a}{4a^2+14ab+49b^2}, \dfrac{ab}{8a^3-343b^3}$.

76. $\dfrac{1}{x+a}, \dfrac{2a}{(x+a)^2}, \dfrac{3}{x-a}, \dfrac{4a}{(x-a)^2}, \dfrac{5a}{x^2-a^2}$.

77. $\dfrac{1}{x-y}, \dfrac{1}{x^2+xy+y^2}, \dfrac{1}{x^2-2xy+y^2}, \dfrac{1}{x^3-y^3}$.

78. $\dfrac{a}{x^2+(b+c)x+bc}, \dfrac{b}{x^2+(c-a)x-ca}, \dfrac{c}{x^2-(a-b)x-ab}$.

79. $\dfrac{1}{x^2-a^2}, \dfrac{1}{x^2-(a+b)x+ab}, \dfrac{1}{x^2+(a-b)x-ab}$.

80. $\dfrac{1}{x^2-4x+3}, \dfrac{1}{x^2+4x-21}, \dfrac{1}{x^2+6x-7}$.

81. $\dfrac{x+3}{x^2-15x+36}, \dfrac{x-3}{x^2-9x-36}, \dfrac{x}{x^2-144}$.

82. $\dfrac{1}{2x^2+3ax-2a^2}, \dfrac{1}{3x^2+8ax+4a^2}, \dfrac{1}{6x^2+ax-2a^2}$.

83. $\dfrac{x+2y}{x^3-8y^3}, \dfrac{1}{x^2+2xy+4y^2}, \dfrac{1}{x^2-4xy+4y^2}$.

84. $\frac{1}{x^3-4x^2+3}, \quad \frac{1}{x^2-2x+1}, \quad \frac{1}{x^2-3x-3}.$

85. $\frac{1}{x^2-y^2}, \quad \frac{1}{x^2+2xy+y^2}, \quad \frac{1}{x^2-xy+y^2}, \quad \frac{x-y}{x^3+y^3}.$

Reduce to their lowest common denominator:

86. $\frac{1}{a^2-5ab-6b^2}, \quad \frac{a-b}{a^3-4a^2b-11ab^2-6b^3}.$

87. $\frac{1}{a^3-3a^2b-10ab^2+24b^3}, \quad \frac{1}{a^3-4a^2b-11ab^2+30b^3}.$

88. $\frac{1}{x^3-x^2-x-2}, \quad \frac{1}{x^3+3x^2+3x+2}.$

89. $\frac{x-y}{x^3+2x^2y+2xy^2+2y^3}, \quad \frac{x^2+y^2}{x^4+x^3y-2y^4}.$

90. $\frac{1}{x^4-6a^2x^2+8a^3x-3a^4}, \quad \frac{1}{x^4-ax^3-3a^2x^2+5a^3x-2a^4}.$

By inspection find the value of

91. $\frac{1}{a-b}+\frac{1}{a+b}.$

92. $\frac{1}{a+b}+\frac{2b}{a^2-b^2}.$

93. $\frac{1}{a-b}-\frac{b}{a^2-b^2}.$

94. $\frac{a+b}{a-b}-\frac{a-b}{a+b}.$

95. $\frac{a}{a-b}+\frac{b}{b-a}.$

96. $\frac{a^2+ab}{a^2b-b^3}-\frac{b}{a^2-ab}.$

97. $\frac{a}{a-b}-\frac{b}{a+b}-\frac{2ab}{a^2-b^2}.$

98. $\frac{a+b}{a-b}+\frac{a-b}{a+b}-\frac{2a^2+2b^2}{a^2-b^2}.$

99. $\frac{1}{x-1}+\frac{1}{x+1}-\frac{2}{x^2-1}.$

100. $\frac{2}{x-3}-\frac{2}{x+3}-\frac{3}{x-2}+\frac{3}{x+2}.$

101. $\frac{a}{a-x}+\frac{a}{a+x}-\frac{a^2+x^2}{a^2-x^2}.$

102. $\frac{x^2+xy+y^2}{x^3-y^3}+\frac{x^2-xy+y^2}{x^3+y^3}+\frac{2y}{x^2-y^2}.$

103. $\frac{1}{x^2+xy}+\frac{1}{xy+y^2}-\frac{1}{xy}.$

104. $\frac{x+y}{x^2-xy+y^2}-\frac{1}{x+y}-\frac{2xy}{x^3+y^3}.$

105. $\frac{1}{x-1}+\frac{3}{x^2+x+1}-\frac{x^2+4x-3}{x^3-1}.$

106. $\frac{1}{x+1}+\frac{1}{x^2-x+1}-\frac{x^2+1}{x^3+1}.$

107. $\dfrac{x-y}{x^2-xy+y^2}+\dfrac{x+y}{x^2+xy+y^2}-\dfrac{2x^3}{x^4+x^2y^2+y^4}$.

108. $\dfrac{a}{(a-b)(x-a)}+\dfrac{b}{(b-a)(x-b)}-\dfrac{b}{x^2-(a+b)x+ab}$.

109. $\dfrac{a^2}{(a-b)(x-a)}+\dfrac{b^2}{(b-a)(x-b)}-\dfrac{bx}{x^2-(a+b)x+ab}$.

110. $\dfrac{1}{x^2-(b+c)x+bc}+\dfrac{2}{x^2-(c+a)x+ca}-\dfrac{3}{x^2-(a+b)x+ab}$.

111. $\dfrac{1}{x^2-5x+6}+\dfrac{1}{x^2+5x-24}$.

112. $\dfrac{1}{x^2-x-156}-\dfrac{1}{x^2+x-132}$.

113. $\dfrac{1}{2x^2-3x-2}+\dfrac{1}{2x^2+3x+1}$.

114. $\dfrac{2}{2x^2+5x-12}-\dfrac{3}{3x^2+10x-6}$.

115. $\dfrac{1}{x^2-4x+4}-\dfrac{2}{x^2-4}+\dfrac{1}{x^2+4x+4}$.

116. $\dfrac{2x+4}{x^2+4x-5}-\dfrac{x+1}{x^2+3x-10}-\dfrac{x-5}{x^2-3x+2}$.

117. $\dfrac{3}{x^2-6x+9}-\dfrac{x}{x^2-x-6}+\dfrac{1}{x+2}$.

118. $\dfrac{3(x+1)}{x^3-4x+3}-\dfrac{2}{x^2-2x+1}-\dfrac{1}{x^2+x-3}$.

119. $\dfrac{(x+y-z)^2}{x^2+y^2-z^2+2xy}+\dfrac{(y+z-x)^2}{y^2+z^2-x^2+2yz}+\dfrac{(z+x-y)^2}{z^2+x^2-y^2+2zx}$.

120. $\dfrac{y+z}{(x-y)(x-z)}+\dfrac{z+x}{(y-z)(y-x)}+\dfrac{x+y}{(z-x)(z-y)}$.

Find by inspection the following products:

121. $\dfrac{ac^2}{4b}\times\dfrac{2b^2}{3a^3c}$.

122. $\dfrac{yz}{x^2}\times\dfrac{zx}{y^2}\times\dfrac{xy}{z^2}$.

123. $\dfrac{ab}{a^2-b^2}\times\dfrac{ab+b^2}{a^2-ab}$.

124. $\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\times\dfrac{ab}{a+b}$.

125. $\dfrac{a-b}{a^2+2ab+b^2}\times\dfrac{a+b}{a^2-2ab+b^2}$.

126. $\dfrac{(x+y)^2}{x^3-y^3}\times\dfrac{(x-y)^2}{x^3+y^3}\times\dfrac{x^4+x^2y^2+y^4}{x^2-y^2}$.

127. $\dfrac{x+y}{x^2+xy+y^2}\times\dfrac{x^2+y^2}{x^3+y^3}\times\dfrac{x^4+x^2y^2+y^4}{x^4+2x^2y^2+y^4}$.

128. $\dfrac{x^2-(b+c)x+bc}{x^2-(b-c)x-bc} \times \dfrac{x^2+(c+a)x+ca}{x^2-(c-a)x-ca}$.

129. $\dfrac{x^2+(a+b)x+ab}{x^2-a^2} \times \dfrac{x^2-(a+b)x+ab}{x^2-b^2}$.

130. $\dfrac{x-a}{x^2-(b-c)x-bc} \times \dfrac{x+b}{x^2-(c+a)x+ca} \times \dfrac{x-c}{x^2-(a-b)x-ab}$.

131. $\dfrac{x^2+x-6}{x^2-4x+3} \times \dfrac{x^2-2x-3}{x^2-x-12} \times \dfrac{x^2-5x+4}{x^2-x-2}$.

132. $\dfrac{2x^2+5x-3}{3x^2+5x-2} \times \dfrac{6x^2+7x-3}{3x^2+7x-6} \times \dfrac{3x^2+4x-4}{4x^2+4x-3}$.

133. $\dfrac{x^2-6x+9}{x^2+3x-28} \times \dfrac{x^2+11x+28}{x^2+2x-15} \times \dfrac{x^2+x-20}{x^2+x-12}$.

134. $\dfrac{x^3-x}{x^3-4x^2+3} \times \dfrac{x^2-3x-3}{x^2+2x+1} \times \dfrac{x^2-x-2}{x^2-4}$.

135. $\dfrac{x^2-y^2-z^2+2yz}{(y+z-x)^2} \times \dfrac{y^2-z^2-x^2+2zx}{(z+x-y)^2} \times \dfrac{z^2-x^2-y^2+2xy}{(x+y-z)^2}$.

136. $\dfrac{x(z-x-y)+yz}{x(z+x-y)-yz} \times \dfrac{y(x-y-z)+zx}{y(x+y-z)-zx} \times \dfrac{z(y-z-x)+xy}{z(y+z-x)-xy}$.

137. $\dfrac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}} \times \dfrac{1-\dfrac{2xy}{(x+y)^2}}{1-\dfrac{x^2+y^2}{(x+y)^2}}$.

138. $\dfrac{\dfrac{x-y}{x^2+xy+y^2}-\dfrac{1}{x-y}}{\dfrac{x+y}{x^2-xy+y^2}-\dfrac{1}{x+y}} \times \dfrac{\dfrac{1}{x^3}-\dfrac{1}{y^3}}{\dfrac{1}{x^3}+\dfrac{1}{y^3}}$.

139. $\dfrac{\dfrac{x}{3y}-1+\dfrac{2y}{3x}}{\dfrac{x}{3y}-1-\dfrac{4y}{3x}} \times \dfrac{\frac{1}{6}\left(\dfrac{x}{y}-1\right)-\dfrac{2y}{x}}{\frac{1}{6}\left(\dfrac{x}{y}+1\right)-\dfrac{y}{x}}$.

140. $\dfrac{\left(1+\dfrac{x}{y}\right)\left(1-\dfrac{x}{z}\right)}{\dfrac{y}{z}-\dfrac{z}{y}} \times \dfrac{\left(1+\dfrac{y}{z}\right)\left(1-\dfrac{y}{x}\right)}{\dfrac{z}{x}-\dfrac{x}{z}} \times \dfrac{\left(1+\dfrac{z}{x}\right)\left(1-\dfrac{z}{y}\right)}{\dfrac{x}{y}-\dfrac{y}{x}}$.

Find the following products:

141. $\dfrac{a^2-b^2-2bc-c^2}{a^3+b^3+c^3-3abc} \times \dfrac{a^2+b^2+c^2-bc-ca-ab}{a^2-2ab+b^2-c^2}$.

142. $\dfrac{2a^2+3ab+b^2}{2a^2-2ab+b^2} \times \dfrac{2a^3-ab^2+b^3}{2a^3+a^2b-2ab^2-b^3}$.

143. $\dfrac{x^2+2x+2}{x^3-x^2-5x+2} \times \dfrac{x^2-3x+1}{x^3-2x-4}$.

144. $\dfrac{x^3+2x^2+x-4}{x^3-x^2-2x-12} \times \dfrac{x^3+x^2+2x-4}{x^3-5x-12}$.

145. $\dfrac{x^3-xy^2-6y^3}{3x^3-5x^2y+xy^2-6y^3} \times \dfrac{3x^3-2x^2y+2xy^2-3y^3}{x^3+x^2y+xy^2-3y^3}$.

146. $\dfrac{x^4+x^3+2x^2-x-3}{x^4-4x^3+8x^2-11x+6} \times \dfrac{x^4-x^3-x^2+4x-12}{x^4-2x^3+2x^2-7x+6}$.

147. $\dfrac{x^4-ax^3-a^3x+a^4}{x^4-3ax^3-2a^2x^2-3a^3x+a^4} \times \dfrac{x^4-ax^3-10a^2x^2-a^3x+a^4}{x^4+ax^3-4a^2x^2+a^3x+a^4}$.

148. $\dfrac{x^2-8-\dfrac{3}{x}}{x^2-15-\dfrac{4}{x}} \times \dfrac{x+4+\dfrac{1}{x}}{x+3+\dfrac{1}{x}}$.

149. $\dfrac{\dfrac{x^2}{6}-\dfrac{x}{3}-\dfrac{1}{3}-\dfrac{1}{2x}}{\dfrac{x^2}{6}-\dfrac{2x}{3}-\dfrac{1}{3}-\dfrac{1}{2x}} \times \dfrac{\frac{1}{2}(x^2+x-1)+\dfrac{1}{x}}{\frac{1}{2}(x^2-x-1)-\dfrac{1}{x}}$.

150. $\dfrac{\dfrac{y}{z}-\dfrac{z}{y}+\dfrac{z}{x}-\dfrac{x}{z}+\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{y}{z}+\dfrac{z}{y}-\dfrac{z}{x}-\dfrac{x}{z}-\dfrac{x}{y}-\dfrac{y}{x}+2} \times \dfrac{\dfrac{1}{z}+\dfrac{1}{y}}{\dfrac{1}{z}-\dfrac{1}{y}}$.

Find by inspection the following quotients:

151. $\dfrac{2a^2b}{3c^3} \div \dfrac{16a}{9c}$.

152. $\dfrac{19a^3b^2y^4z^5}{20c^4x^4} \div \dfrac{133a^2by^5z^6}{100c^5x^3}$.

153. $\left(\dfrac{a}{b}-\dfrac{b}{a}\right) \div \dfrac{a-b}{ab}$.

154. $\dfrac{a+b}{a^2-2ab+b^2} \div \dfrac{a^2+2ab+b^2}{a-b}$.

155. $\dfrac{a^2-ab+b^2}{a^3-b^3} \div \dfrac{a^3+b^3}{a^2+ab+b^2}$.

156. $\left(ax^2-\dfrac{1}{a}\right) \div \left(x-\dfrac{1}{a}\right)$.

157. $\left(\dfrac{x^3}{a^3}+\dfrac{a^3}{x^3}\right) \div \left(\dfrac{x^2}{a^2}-1+\dfrac{a^2}{x^2}\right)$.

158. $\left(x^2+1+\dfrac{1}{x^2}\right) \div \left(x+1+\dfrac{1}{x}\right)$.

159. $\left(x+\dfrac{y^3}{x^2}\right) \div \left(x-y+\dfrac{y^2}{x}\right)$.

160. $\left(\dfrac{x^2}{y^2}-\dfrac{y}{x}\right) \div \left(\dfrac{x}{y}+1+\dfrac{y}{x}\right)$.

161. $\left(\dfrac{x^2}{y^2}+1+\dfrac{y^2}{x^2}\right) \div \left(\dfrac{x}{y}-1+\dfrac{y}{x}\right)$.

162. $\dfrac{x^3+y^3}{x^4+x^2y^2+y^4} \div \dfrac{x^2-y^2}{x^3-y^3}$.

163. $\dfrac{(x+y)^2-z^2}{x^2-(y-z)^2} \div \dfrac{x^2-(y+z)^2}{(x-y)^2-z^2}$.

164. $\dfrac{x^2-(b-c)x-bc}{x^2+(c+a)x+ca} \div \dfrac{x^2-(a+b)x+ab}{x^2-a^2}$.

165. $\dfrac{x^2+2x-3}{x^2+3x-4} \div \dfrac{x^2+x-6}{x^2+2x-8}$.

166. $\dfrac{x^2+12x+36}{x^2+x-30} \div \dfrac{x^2-36}{x^2-11x+30}$.

167. $\dfrac{x^2-3xy+9y^2}{x^2+3xy+9y^2} \div \dfrac{x^3+27y^3}{x^3-27y^3}$.

168. $\dfrac{x^2-y^2-z^2+2yz}{4y^2z^2-(x^2-y^2-z^2)^2} \div \dfrac{x-y-z}{x^2-y^2-z^2-2yz}$.

169. $\dfrac{x^2+y^2+z^2-yz-zx-xy}{x^2+y^2-z^2-2xy} \div \dfrac{(y-z)^2+(z-x)^2+(x-y)^2}{x^2-y^2-z^2+2yz}$.

170. $\dfrac{\frac{x}{2}+2+\frac{2}{x}}{\frac{x}{2}-1-\frac{4}{x}} \div \dfrac{\frac{x}{3}+1+\frac{2}{3x}}{\frac{x}{3}-1-\frac{4}{3x}}$.

171. $\dfrac{\frac{x}{4y}-\frac{4y}{x}}{\frac{x}{3y}-\frac{3y}{x}} \div \dfrac{\frac{x}{4y}-2+\frac{4y}{x}}{\frac{x}{3y}+2+\frac{3y}{x}}$.

172. $\dfrac{\frac{x^4}{a^4}+1+\frac{a^4}{x^4}}{\frac{x^3}{a^3}-\frac{a^3}{x^3}} \div \dfrac{\frac{x^3}{a^3}+\frac{a^3}{x^3}}{\frac{x^2}{a^2}-\frac{a^2}{x^2}}$.

Simplify

173. $\dfrac{1}{1-\dfrac{1}{1+\frac{1}{x}}}$.

174. $1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1-\frac{1}{x}}}}$.

175. $1+\dfrac{x}{1-\dfrac{x}{1-\frac{1}{x}}}$.

176. $\dfrac{1}{1-\dfrac{x}{1-\dfrac{x}{1-x+\frac{x^2}{1+x}}}}$.

177. $\dfrac{x-2-\dfrac{24}{x+3}}{x-4-\dfrac{30}{x+3}}$.

178. $\dfrac{x+3+\dfrac{1}{x-4}}{x+4+\dfrac{9}{x-5}}$.

179. $\dfrac{x^2+2x+4+\dfrac{8}{x-2}}{x^2-2x+4-\dfrac{8}{x+2}}$.

180. $\dfrac{x^2-3x+2+\dfrac{4x^2}{x+1}}{x^2+3x+2+\dfrac{4}{x-1}}$.

181. $\dfrac{4}{x+1}-\dfrac{3x-1}{x^2+\frac{2}{3}x-\frac{1}{3}}$.

182. $$\frac{y+z}{\left(\frac{y}{x}-1\right)\left(\frac{z}{x}-1\right)}+\frac{z+x}{\left(\frac{z}{y}-1\right)\left(\frac{x}{y}-1\right)}+\frac{x+y}{\left(\frac{x}{z}-1\right)\left(\frac{y}{z}-1\right)}.$$

183. $$\frac{\frac{1}{y}+\frac{1}{z}}{\left(1-\frac{x}{y}\right)\left(1-\frac{x}{z}\right)}+\frac{\frac{1}{z}+\frac{1}{x}}{\left(1-\frac{y}{z}\right)\left(1-\frac{y}{x}\right)}+\frac{\frac{1}{x}+\frac{1}{y}}{\left(1-\frac{z}{x}\right)\left(1-\frac{z}{y}\right)}.$$

Prove that

184. $$\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}=0.$$

185. $$\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}=0.$$

186. $$\frac{b+c}{(a-b)(a-c)}+\frac{c+a}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)}=0.$$

187. $$\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}=1.$$

188. $$\frac{bc}{(a-b)(a-c)}+\frac{ca}{(b-c)(b-a)}+\frac{ab}{(c-a)(c-b)}=1.$$

189. $$\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-c)(b-a)}+\frac{c^3}{(c-a)(c-b)}=a+b+c.$$

If $2s=a+b+c$, prove that

190. $$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

191. $$\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}+\frac{1}{(s-a)(s-b)}=\frac{s}{(s-a)(s-b)(s-c)}.$$

192. $$\frac{b+c}{(s-b)(s-c)}+\frac{c+a}{(s-c)(s-a)}+\frac{a+b}{(s-a)(s-b)}=\frac{a^2+b^2+c^2}{(s-a)(s-b)(s-c)}.$$

193. $$\frac{s-a}{s(s-b)(s-c)}+\frac{s-b}{s(s-c)(s-a)}+\frac{s-c}{s(s-a)(s-b)}+\frac{s}{(s-a)(s-b)(s-c)}=\frac{a^2+b^2+c^2}{s(s-a)(s-b)(s-c)}.$$

194. $$\frac{s-a}{(s-b)(s-c)}+\frac{s-b}{(s-c)(s-a)}+\frac{s-c}{(s-a)(s-b)}+\frac{s^2}{(s-a)(s-b)(s-c)}=\frac{a^2+b^2+c^2}{(s-a)(s-b)(s-c)}.$$

195. $$\frac{1}{s(s-b)(s-c)}+\frac{1}{s(s-c)(s-a)}+\frac{1}{s(s-a)(s-b)}=\frac{1}{(s-a)(s-b)(s-c)}.$$

196. If the sum of two fractions is unity, shew that their difference is equal to the difference of their squares.

197. If the difference of two fractions is $\frac{m}{n}$, shew that m times their sum is n times the difference of their squares.

198. If $a-\frac{1}{a}=1$, prove that $a^2+\frac{1}{a^2}=3$ and $a^3-\frac{1}{a^3}=4$.

199. If $a^2+\frac{1}{a^2}=2$, prove that $a^{2n}+\frac{1}{a^{2n}}=2$.

200. If $a^3-\frac{1}{a^3}=3\left(a-\frac{1}{a}\right)$, prove that $a^2=1$.

CHAPTER X.

INVOLUTION, EVOLUTION.

74. The process of multiplying a quantity continually by itself and so obtaining its successive powers is called **Involution**; involution is thus a branch of multiplication and requires no special consideration. In the chapter on multiplication we have obtained the following results in involution:

$$(a+b)^2 = a^2 + 2ab + b^2;$$
$$(a-b)^2 = a^2 - 2ab + b^2;$$
$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2bc + 2ca + 2ab;$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3;$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3;$$
$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3a^2(b+c) + 3b^2(c+a) + 3c^2(a+b) + 6abc;$$

and in the same manner any other power of an algebraical expression can be found by successive multiplication. *All squares and cubes of binomial or trinomial expressions should be written down by inspection from* the *formulae* of this Article and should not be found by actual work.

Example 1. Find the square of $x-1+\frac{1}{x}$.

$$\left(x+\frac{1}{x}-1\right)^2 = \left(x+\frac{1}{x}\right)^2 - 2\left(x+\frac{1}{x}\right)+1$$
$$= x^2+\frac{1}{x^2} - 2\left(x+\frac{1}{x}\right)+3.$$

Example 2. Find the square of $\sqrt{2x-y}+\sqrt{2y-x}$.

$$(\sqrt{2x-y}+\sqrt{2y-x})^2 = 2x-y+2\sqrt{(2x-y)(2y-x)}+2y-x$$
$$= x+y+2\sqrt{5xy-2(x^2+y^2)}.$$

Example 3. Find the cube of $\frac{x}{a}-\frac{a}{x}$.

$$\left(\frac{x}{a}-\frac{a}{x}\right)^3=\frac{x^3}{a^3}-3\frac{x^2}{a^2}\times\frac{a}{x}+3\frac{x}{a}\times\frac{a^2}{x^2}-\frac{a^3}{x^3}$$

$$=\frac{x^3}{a^3}-\frac{a^3}{x^3}-3\left(\frac{x}{a}-\frac{a}{x}\right).$$

Example 4. Find the square of $x^2-2x+\frac{3}{x}$.

$$\left(x^2-2x+\frac{3}{x}\right)^2=x^4+4x^2+\frac{9}{x^2}-2\times 2x\times\frac{3}{x}+2\times\frac{3}{x}\times x^2-2\times x^2\times 2x$$

$$=x^4-4x^3+4x^2+6x-12+\frac{9}{x^2}.$$

75. Evolution is the reverse of involution and is the process by which we discover the quantity which when raised to a certain power will produce a given quantity; evolution is therefore the extraction of roots.

We have proved (Art. 70) that any power of a fraction is obtained by raising both numerator and denominator to that power; it follows therefore that any root of a fraction is obtained by extracting the root of both numerator and denominator.

76. By the Law of Signs we know that all powers of a positive quantity are positive and that powers of a negative quantity are alternately positive and negative; hence all even powers of any quantity, positive or negative, are positive, and all odd powers have the same sign as the original quantity. These statements in involution lead to the corresponding statements in evolution that *an even root of a positive quantity is either positive or negative, that there can be no even root of a negative quantity, and that an odd root of a quantity has the sign of the original quantity.*

SQUARE ROOT.

77. A positive quantity has two square roots differing only in sign, and either of these may be regarded as the square root. This follows from the law of signs as already explained, or may at once be separately proved, for

$$a^2=+a\times+a \text{ or } =-a\times-a;$$

and the fact is expressed by the statement that

$$\sqrt{a^2} = \pm a,$$

the double sign indicating that either the positive or negative sign may be taken. In practice it is usual to omit the negative sign, but the double sign is always to be understood when not expressed. A negative quantity has no square root.

78. The square root of a simple expression can always be written down by inspection; thus the square root of a^2b^2 is $\pm ab$, of $81x^2y^4z^6$ is $\pm 9xy^2z^3$ and of $\frac{49x^4}{144c^6}$ is $\pm \frac{7x^2}{12c^3}$.

In the case of either a simple or compound expression the square root of a numerical coefficient, if not apparent on inspection, must be found by arithmetic.

79. The square root of a trinomial expression which is a complete square may also always be written down by inspection. For we have $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$, so that the square root of the binomial $a^2 + 2ab + b^2$ is the binomial $a + b$, and of $a^2 - 2ab + b^2$ is $a - b$. The square root is thus found by arranging the given trinomial in descending powers of some letter, by then finding the square roots of the first and last terms and finally by connecting these square roots by the + or − sign according as the sign of the middle term is + or −. Thus the square root of $9x^6 + 24x^3 + 16$ is $3x^3 + 4$, of $25x^4 - 20x^2y^4 + 4y^8$ is $5x^2 - 2y^4$, and of $\frac{4x^2}{9a^2} - 2 + \frac{9a^2}{4x^2}$ is $\frac{2x}{3a} - \frac{3a}{2x}$.

In each of these examples it should be observed that the double sign is understood and, to be strictly correct, ought to be introduced; thus the square root of $a^2 + 2ab + b^2$ is $a + b$ or $-a - b$, of $a^2 - 2ab + b^2$ is $a - b$ or $-a + b$, of $9x^6 + 24x^3 + 16$ is $3x^3 + 4$ or $-3x^3 - 4$, of $25x^4 - 20x^2y^4 + 4y^8$ is $5x^2 - 2y^4$ or $-5x^2 + 2y^4$, and of $\frac{4x^2}{9a^2} - 2 + \frac{9a^2}{4x^2}$ is $\frac{2x}{3a} - \frac{3a}{2x}$ or $-\frac{2x}{3a} + \frac{3a}{2x}$.

80. Also the square root of any expression which is a complete square containing only two powers of a letter and arranged in descending powers of that letter can be written down by inspection. Thus the expression

$$a^2 + b^2 + c^2 + 2bc + 2ca + 2ab$$

may be written

$$a^2 + 2a(b+c) + b^2 + 2bc + c^2,$$

i.e.

$$a^2 + 2a(b+c) + (b+c)^2,$$

and is reduced to the form already considered, so that its square root is

$$a + (b+c),$$

i.e.

$$a + b + c.$$

So also the expression

$$x^6 + 4a^2y^2 + 25a^4 - 20a^3y + 10a^2x^3 - 4ax^3y$$

may be written

$$x^6 - (4ay - 10a^2)\,x^3 + 4a^2y^2 - 20a^3y + 25a^4,$$

i.e.

$$x^6 - 2(2ay - 5a^2)\,x^3 + (2ay - 5a^2)^2;$$

and its square root is therefore

$$x^3 - (2ay - 5a^2),$$

i.e.

$$x^3 - 2ay + 5a^2.$$

When the given expression is a multinomial containing more than two powers of any letter, and it is not immediately apparent how the terms should be arranged so as to bring the expression into the typical form of a complete square, another method must be employed; and we proceed to investigate a general rule for extracting the square root of any algebraical expression. It should however be remembered that the rule is to be applied charily and that *a solution by inspection is always to be preferred.*

81. In order to obtain a rule for extracting the square root of any algebraical expression N we proceed to examine how the square of an algebraical expression is formed and how from this square, if given, the original expression may be deduced.

Let the given expression N be equivalent to the complete square $(a+b)^2$, i.e. $a^2 + 2ab + b^2$, arranged in descending powers of the letter a. Extracting the square root of the first term a^2 we obtain a, the first term of the required root; and subtracting a^2 from the given expression N we have

$$N - a^2 = 2ab + b^2.$$

Then to obtain b, the remaining part of the root, we must obviously divide the first term $2ab$ of the remainder by $2a$, twice the part of the root already found; adding this new term b to $2a$ we get $2a + b$, and dividing $N - a^2$, i.e. $2ab + b^2$, i.e. $(2a+b)\,b$, by $2a + b$, we have the quotient b and no remainder, and the required square root $a + b$ has been found.

If the required square root of N be a trinomial $a+b+c$, then

$$N=(a+b+c)^2=(a+b)^2+2(a+b)c+c^2;$$

so that $$N-(a+b)^2=2(a+b)c+c^2.$$

Thus we first obtain the part $a+b$ of the root by the method already explained; then to obtain the remaining part c we must treat the part $a+b$ of the root already found as we formerly treated a. We must divide the first term $2(a+b)c$ of the remainder by $2(a+b)$, i.e. by twice the part of the root already found; adding the quotient c to $2(a+b)$ we get $2(a+b)+c$, and dividing $N-(a+b)^2$, i.e. $2(a+b)c+c^2$, i.e. $\{2(a+b)+c\}c$, by $2(a+b)+c$ we have the quotient c and no remainder, and the required square root $a+b+c$ has been found. If there were more terms in the square root we should proceed to treat $a+b+c$ as we formerly treated a and $a+b$ in succession, and the process must be continued until there is no remainder.

The operations we have described may be conveniently arranged thus:

$$\begin{array}{rl} & a^2+2ab+b^2\,(\,a+b \\ & a^2 \\ \hline 2a+b\,) & 2ab+b^2 \\ & 2ab+b^2 \\ \hline \end{array}$$

$$\begin{array}{rl} & a^2+2ab+2ca+b^2+2bc+c^2\,(\,a+b+c \\ & a^2 \\ \hline 2a+b\,) & 2ab\quad\;+b^2 \\ & 2ab\quad\;+b^2 \\ \hline 2a+2b+c\,) & 2ca\quad+2bc+c^2 \\ & 2ca\quad+2bc+c^2 \\ \hline \end{array}$$

and so on.

82. The square root of an algebraical expression is hence obtained by the following rule:

Arrange the expression according to descending powers of some letter. Find the square root of the first term and put down the result as the first term of the root; subtract its square from the given expression and bring down the remainder. Double the first term of the root and make this the first term of a divisor; divide the first term of the remainder by this first term of the divisor, make the quotient the second term of the root and add it to the first term of the divisor, thus completing the divisor. Multiply the divisor by the second term of the root and subtract the result from

the first remainder. Treat the two terms of the root already found as one and proceed as before; continue the process until there is no remainder.

Example 1. Extract the square root of

$$x^6-4ax^5+2a^2x^4+10a^3x^3-11a^4x^2-6a^5x+9a^6.$$

$$\begin{array}{l|l}
x^6-4ax^5+2a^2x^4+10a^3x^3-11a^4x^2-6a^5x+9a^6 & x^3-2ax^2-a^2x+3a^3 \\
x^6 & \\
\hline
2x^3-2ax^2 \;)\; -4ax^5+2a^2x^4 & \\
\quad -4ax^5+4a^2x^4 & \\
\hline
2x^3-4ax^2-a^2x \;)\; -2a^2x^4+10a^3x^3-11a^4x^2 & \\
\quad -2a^2x^4+\;4a^3x^3+\;\;a^4x^2 & \\
\hline
2x^3-4ax^2-2a^2x+3a^3 \;)\; 6a^3x^3-12a^4x^2-6a^5x+9a^6 & \\
\quad 6a^3x^3-12a^4x^2-6a^5x+9a^6 & \\
\end{array}$$

Example 2. Find the square root of $\frac{x^2}{4y^2}-x+y^2+\frac{2}{y}-\frac{4y}{x}+\frac{4}{x^2}$.

$$\begin{array}{l|l}
\frac{x^2}{4y^2}-x+y^2+\frac{2}{y}-\frac{4y}{x}+\frac{4}{x^2} & \frac{x}{2y}-y+\frac{2}{x} \\
\frac{x^2}{4y^2} & \\
\hline
\frac{x}{y}-y \;)\; -x+y^2 & \\
\quad -x+y^2 & \\
\hline
\frac{x}{y}-2y+\frac{2}{x} \;)\; \frac{2}{y}-\frac{4y}{x}+\frac{4}{x^2} & \\
\quad \frac{2}{y}-\frac{4y}{x}+\frac{4}{x^2} & \\
\end{array}$$

Example 3. Extract the square root of

$$1+2x-5x^3-\frac{15}{4}x^4+2x^5+4x^6.$$

$$\begin{array}{l|l}
1+2x-5x^3-\frac{15}{4}x^4+2x^5+4x^6 & 1+x-\frac{x^2}{2}-2x^3 \\
1 & \\
\hline
2+x \;)\; 2x & \\
\quad 2x+x^2 & \\
\hline
2+2x-\frac{x^2}{2} \;)\; -x^2-5x^3-\frac{15}{4}x^4 & \\
\quad -x^2-\;x^3+\;\;\frac{x^4}{4} & \\
\hline
2+2x-x^2-2x^3 \;)\; -4x^3-\;\;4x^4+2x^5+4x^6 & \\
\quad -4x^3-\;\;4x^4+2x^5+4x^6 & \\
\end{array}$$

Sometimes, as in this example, the expression is arranged in ascending, and not in descending, powers of some letter; the arrangement in descending powers is more usual, but it is a matter of indifference, provided one method or the other be adopted.

Example 4. Find the condition that the quadratic expression ax^2+bx+c may be a perfect square.

The expression will obviously be a perfect square if in the process of extracting the square root the remainder vanish.

$$\begin{array}{r|l} ax^2+bx+c & \left(\sqrt{a}x+\dfrac{b}{2\sqrt{a}}\right. \\ ax^2 & \\ \hline 2\sqrt{a}x+\dfrac{b}{2\sqrt{a}}\Big) & bx+c \\ & bx+\dfrac{b^2}{4a} \\ \hline & c-\dfrac{b^2}{4a} \end{array}$$

Thus the remainder $c-\dfrac{b^2}{4a}$ must vanish; hence the condition required is $b^2=4ac$.

An expression is a *perfect* or *complete* or *exact* square when its square root can be extracted exactly, and it may be proved to be a perfect square either by transforming it into the known form of a square or by extracting its square root exactly. Also just as in arithmetic we can extract the square root of any number correct to as many decimal places as we please, so in algebra when an expression is not a perfect square, its square root may still by the ordinary process be extracted partially to as many terms as we please.

CUBE ROOT.

83. The cube root of a positive quantity is positive and of a negative quantity is negative. This follows from the law of signs as stated in Art. 76 or may at once be proved separately; for $a^3=a\times a\times a$, and $-a^3=a^2\times -a=-a\times -a\times -a$, so that $\sqrt[3]{a^3}=a$ and $\sqrt[3]{-a^3}=-a$.

84. The cube root of a simple expression can be written down at once by inspection; thus the cube root of a^3b^3 is ab, of $-125x^3y^6z^9$ is $-5xy^2z^3$, and of $\dfrac{8x^6}{27c^3}$ is $\dfrac{2x^2}{3c}$. In the case of either a simple or compound expression the cube root of a numerical coefficient, when not apparent on inspection, must be found by arithmetic.

85. The cube root of a quadrinomial, i.e. of a polynomial of four terms, which is a complete cube, can also be written down by inspection. For we know that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3;$$

so that the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$ is $a+b$, and of $a^3 - 3a^2b + 3ab^2 - b^3$ is $a-b$.

The cube root is hence found by arranging the given expression in descending powers of some letter, by then finding the cube roots of the first and last terms, and finally by adding these cube roots together.

Thus the cube root of $x^3 + 6x^2 + 12x + 8$ is $x+2$, of

$x^3 - 6x^2 + 12x - 8$ is $x-2$, of $-x^3 - 6x^2 - 12x - 8$ is $-x-2$,

and of $-x^3 + 6x^2 - 12x + 8$ is $-x+2$.

86. Also the cube root of any expression which is a complete cube, containing only three powers of a letter and arranged in descending powers of that letter, can similarly be written down by inspection. Thus the expression

$$a^3 + b^3 + c^3 + 3a^2(b+c) + 3b^2(c+a) + 3c^2(a+b) + 6abc$$

may be written

$$a^3 + 3a^2(b+c) + 3a(b^2 + 2bc + c^2) + b^3 + 3b^2c + 3bc^2 + c^3,$$

i.e. $a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3$;

and is thus reduced to the form already considered, so that its cube root is

$$a + (b+c),$$

i.e. $$a + b + c.$$

So also the expression

$$x^6 - 3ax^4y + 6a^2x^4 + 3a^2x^2y^2 - 12a^3x^2y + 12a^4x^2 - a^3y^3 + 6a^4y^2 - 12a^5y + 8a^6$$

may be written

$$x^6 - 3x^4(ay - 2a^2) + 3x^2(a^2y^2 - 4a^3y + 4a^4) - (a^3y^3 - 6a^4y^2 + 12a^5y - 8a^6),$$

i.e. $x^6 - 3x^4(ay - 2a^2) + 3x^2(ay - 2a^2)^2 - (ay - 2a^2)^3$;

and its cube root is therefore

$$x^2 - (ay - 2a^2),$$

i.e. $$x^2 - ay + 2a^2.$$

When however the given expression is not of this shape and contains more than three powers of any letter, and it does not immediately appear how it can be arranged in the typical form of a complete cube, another method must be employed. We proceed therefore to investigate a general rule for extracting the cube root of any algebraical expression.

87. Let us examine how the cube of an algebraical expression is formed and how from this cube, when given, the required cube root can be deduced.

Let N denote the given expression which is a complete cube of the form $(a+b)^3$, i.e.

$$a^3 + 3a^2b + 3ab^2 + b^3,$$

arranged in descending powers of the letter a.

Extracting the cube root of the first term a^3 we obtain a, the first term of the required root; and subtracting a^3 from the given expression N we have

$$N - a^3 = 3a^2b + 3ab^2 + b^3.$$

Then to get b, the remaining part of the root, we must obviously divide the first term $3a^2b$ of the remainder by $3a^2$, thrice the square of the part of the root already found; to $3a^2$ we must now add $3ab$ and b^2, which together form the sum $3a^2 + 3ab + b^2$. Dividing by this expression we have the quotient b and no remainder, and the required cube root has been found.

If the required cube root of N be a trinomial $a+b+c$, then

$$\begin{aligned} N &= (a+b+c)^3 \\ &= (a+b)^3 + 3(a+b)^2 c + 3(a+b)c^2 + c^3, \end{aligned}$$

so that $$N - (a+b)^3 = 3(a+b)^2 c + 3(a+b)c^2 + c^3.$$

Hence we first obtain the part $a+b$ of the root by the method already explained; then to obtain the remaining part c we must treat the part $a+b$ of the root as we formerly treated a, viz. we must divide the first term $3(a+b)^2 c$ of the remainder by $3(a+b)^2$, thrice the square of the part of the root already found; then to $3(a+b)^2$ we must add $3(a+b)c$ and c^2, thus forming the sum $3(a+b)^2 + 3(a+b)c + c^2$; and dividing by this expression we have

the quotient c and no remainder, and the required cube root $a+b+c$ has been found.

If there were more terms in the cube root we should now proceed to treat $a+b+c$ as we formerly treated a and $a+b$ in succession, and the process must be continued until there is no remainder.

The operations we have described may be conveniently arranged thus:

$$\begin{array}{r|l} a^3+3a^2b+3ab^2+b^3 & a+b \\ a^3 & \\ \hline 3a^2+3ab+b^2 \;)\; 3a^2b+3ab^2+b^3 & \\ 3a^2b+3ab^2+b^3 & \\ \hline \end{array}$$

$$\begin{array}{r|l} a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3 & a+b+c \\ a^3 & \\ \hline 3a^2+3ab+b^2)\,3a^2b+3ab^2+b^3 & \\ 3a^2b+3ab^2+b^3 & \\ \hline 3(a+b)^2+3(a+b)c+c^2 & \\ =3a^2+6ab+3ac+3b^2+3bc+c^2\;) \; 3a^2c+6abc+3ac^2+3b^2c+3bc^2+c^3 & \\ 3a^2c+6abc+3ac^2+3b^2c+3bc^2+c^3 & \\ \hline \end{array}$$

The order in which the terms of the given expression are to be brought down is indicated by the order of terms in the divisor, the terms in each divisor and dividend being arranged in descending powers of the letters successively.

88. We have thus proved that the cube root of an algebraical expression may be found by the following rule:

Arrange the expression according to descending powers of some letter. Find the cube root of the first term and put down the result as the first term of the root; subtract its cube from the given expression and bring down the remainder. Treble the square of the first term of the root and make this the first term of a divisor; divide the first term of the remainder by this first term of the divisor, make the quotient the second term of the root, add thrice the product of the quotient and first term of the root and the square of the quotient to the first term of the divisor already found, thus completing the divisor. Multiply the divisor by the second term of the root, and subtract the result from the first remainder. Treat the two terms of the root already found as one and proceed as before; continue the process until there is no remainder.

Example 1. Extract the cube root of $x^6-3ax^5+9a^2x^4-13a^3x^3+18a^4x^2-12a^5x+8a^6$.

$$
\begin{array}{l|l}
x^6-3ax^5+9a^2x^4-13a^3x^3+18a^4x^2-12a^5x+8a^6 & x^2-ax+2a^2 \\
\underline{x^6} & \\
3(x^2)^2+3x^2(-ax)+(-ax)^2 & \\
=3x^4-3ax^3+a^2x^2 & -3ax^5+9a^2x^4-13a^3x^3 \\
 & \underline{-3ax^5+3a^2x^4-\quad a^3x^3} \\
3(x^2-ax)^2+3(x^2-ax)(2a^2)+(2a^2)^2 & \\
=3x^4-6ax^3+9a^2x^2-6a^3x+4a^4 & 6a^2x^4-12a^3x^3+18a^4x^2-12a^5x+8a^6 \\
 & \underline{6a^2x^4-12a^3x^3+18a^4x^2-12a^5x+8a^6}
\end{array}
$$

Example 2. Find the cube root of $\frac{x^3}{8y^3}-\frac{3x^3}{4y^2}+\frac{3x^3}{2y}-x^3+\frac{3x}{2y}-6x+6xy+6\frac{y}{x}-12\frac{y^2}{x}+8\frac{y^3}{x^3}$.

$$
\begin{array}{l|l}
\frac{x^3}{8y^3}-\frac{3x^3}{4y^2}+\frac{3x^3}{2y}-x^3+\frac{3x}{2y}-6x+6xy+6\frac{y}{x}-12\frac{y^2}{x}+8\frac{y^3}{x^3} & \frac{x}{2y}-x+\frac{2y}{x} \\
\underline{\frac{x^3}{8y^3}} & \\
3\left(\frac{x}{2y}\right)^2+3\left(\frac{x}{2y}\right)(-x)+(-x)^2 & \\
=\frac{3x^2}{4y^2}-\frac{3x^2}{2y}+x^2 & -\frac{3x^3}{4y^2}+\frac{3x^3}{2y}-x^3 \\
 & \underline{-\frac{3x^3}{4y^2}+\frac{3x^3}{2y}-x^3} \\
3\left(\frac{x}{2y}-x\right)^2+3\left(\frac{x}{2y}-x\right)\left(\frac{2y}{x}\right)+\left(\frac{2y}{x}\right)^2 & \\
=\frac{3x^2}{4y^2}-\frac{3x^2}{y}+3x^2+3-6y+\frac{4y^2}{x^2} & \frac{3x}{2y}-6x+6xy+6\frac{y}{x}-12\frac{y^2}{x}+8\frac{y^3}{x^3} \\
 & \underline{\frac{3x}{2y}-6x+6xy+6\frac{y}{x}-12\frac{y^2}{x}+8\frac{y^3}{x^3}}
\end{array}
$$

It will be noticed that the rules for the extraction of square root and cube root in arithmetic follow at once from the algebraical methods which have just been explained.

OTHER ROOTS.

89. If m and n be any positive integers it is easily seen that

$$(a^m)^n = a^{mn}.$$

For $(a^m)^n$ is a product containing the factor a^m repeated n times,

$$= a^m \times a^m \times a^m \times \ldots\ldots \text{to } n \text{ factors}$$

$$= a^{m+m+m+\ldots\ldots \text{ to } n \text{ terms}}$$

$$= a^{mn};$$

and similarly also $(a^n)^m = a^{mn}$.

Thus $a^4 = (a^2)^2$, $a^6 = (a^3)^2$ or $= (a^2)^3$, $a^8 = (a^4)^2$ or $= (a^2)^4$, $a^9 = (a^3)^3$, and so on; hence the 4th root of a quantity is the square root of the square root; the 6th root is the square root of the cube root or the cube root of the square root; the 8th root is the square root of the 4th root or the 4th root of the square root, i.e. is the square root of the square root of the square root; the 9th root is the cube root of the cube root, and so on. Theoretically the nth root of an algebraical expression can be found by the methods of this chapter when n is the product of powers of 2 and 3.

Example 1. Find the 4th root of

$$x^4 + \frac{1}{x^4} - 4\left(x^2 + \frac{1}{x^2}\right) + 6.$$

The expression may be written in the form

$$x^4 + 2 + \frac{1}{x^4} - 4\left(x^2 + \frac{1}{x^2}\right) + 4,$$

i.e.
$$\left(x^2 + \frac{1}{x^2}\right)^2 - 4\left(x^2 + \frac{1}{x^2}\right) + 4,$$

i.e.
$$\left(x^2 + \frac{1}{x^2} - 2\right)^2,$$

so that its square root is

$$x^2 + \frac{1}{x^2} - 2;$$

and
$$x^2 - 2 + \frac{1}{x^2} = \left(x - \frac{1}{x}\right)^2,$$

so that the square root of this expression is $x-\frac{1}{x}$, which is the 4th root required.

Or at once: $$x^4+\frac{1}{x^4}-4\left(x^2+\frac{1}{x^2}\right)+6$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}\right)+4$$

$$=\left(x^2+\frac{1}{x^2}-2\right)^2=\left(x-\frac{1}{x}\right)^4;$$

so that the 4th root required is $x-\frac{1}{x}$.

Example 2. Find the 6th root of

$$64a^6-192a^5b+240a^4b^2-160a^3b^3+60a^2b^4-12ab^5+b^6.$$

By the ordinary method the cube root is found to be

$$4a^2-4ab+b^2,$$

i.e. $$(2a-b)^2;$$

and the square root of this is $2a-b$, which is therefore the 6th root required.

Example 3. Find the 8th root of $6561x^8$ and the 9th root of $19683x^9y^9z^{18}$.

By arithmetic the square root of 6561 is 81, of 81 is 9 and of 9 is 3, so that the 8th root required is $3x$.

Also by arithmetic the cube root of 19683 is 27 and the cube root of 27 is 3, so that the 9th root required is $3xyz^2$.

EXAMPLES.

Write down the square of

1. $a^2x^4-\frac{y^3}{a}$.
2. $1-a+a^2$.
3. $\frac{x}{a}+1-\frac{a}{x}$.
4. $2x-\frac{y}{2}+4z$.
5. $\frac{x}{y}-\frac{y}{z}+\frac{z}{x}$.
6. $\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}$.
7. $(1-x+x^2)(1+x+x^2)$.
8. $(1-\sqrt{3}x+x^2)(1+\sqrt{3}x+x^2)$.
9. $\sqrt{2a+x}-\sqrt{a-x}$.
10. $\sqrt{x+y-a}+\sqrt{x-y+a}$.

Write down the cube of

11. $a+\frac{1}{a}$. 12. $1+2a$. 13. $2x-\frac{y}{3}$.

14. $3x^2-4y$. 15. $1+a-a^2$. 16. $(1+a)(1-a+a^2)$.

17. $(1-x+x^2)(1+x+x^2)$. 18. $(1-\sqrt{2}x+x^2)(1+\sqrt{2}x+x^2)$.

19. $1-2x+x^2$. 20. $x-1-\frac{1}{x}$.

Simplify

21. $(a+2b+3c)^2 \pm (a-2b+3c)^2$. 22. $(a+b)^3 \pm (a-b)^3$.

23. $(a+b)^4 \pm (a-b)^4$. 24. $(a+b)^6 \pm (a-b)^6$.

25. $(a+b+c+d)^2 \pm (a+b+c-d)^2$.

26. Find the fourth power of $(x+a)(x-a)$.

27. Find the fourth power of $\frac{x}{a}+\frac{a}{x}$.

28. Find the fifth power of $1-x$.

29. Find the sixth power of $x-\frac{1}{x}$.

30. Find the eighth power of $x+y$.

Find by inspection the square root of

31. $36a^4$. 32. $144x^6y^8z^{10}$. 33. $\frac{121x^2}{169y^4}$.

34. $9a^2+42ab+49b^2$. 35. $16a^4b^4-40a^2b^2c^2+25c^4$.

36. $100x^2+220xy+121y^2$.

37. $(x^2-x-2)(x^2-2x-3)(x^2-5x+6)$.

38. $\frac{4x^2-36x+81}{4x^2-60x+225}$. 39. $\frac{9a^2x^2-24abx+16b^2}{9a^2x^2+24abx+16b^2}$.

40. $\frac{x^2-4x+4}{x^4-8x^2+16}$. 41. $\frac{289x^2+68xy+4y^2}{361x^4+114x^2y^2+9y^4}$.

42. $\frac{a^2x^2}{b^2y^2}-2+\frac{b^2y^2}{a^2x^2}$. 43. $\frac{x^2}{4}+\frac{2}{3}x+\frac{4}{9}$.

44. $\frac{x^2}{9}-\frac{ax}{3}+\frac{a^2}{4}$. 45. $16a^2-2ab+\frac{b^2}{16}$.

46. $x^2+y^2-2ax+2ay-2xy+a^2$.

47. $\frac{a^2y^4}{4}-2a^3y^3+ax^3y^2+4a^4y^2-4a^2x^3y+x^6$.

48. $9a^2+\frac{4b^2}{9}+\frac{c^2}{4}-\frac{2bc}{3}+3ca-4ab.$

49. $x^4-2x^3-(2a-1)\,x^2+2ax+a^2.$

50. $x^4+y^2z^2+\frac{y^6}{z^2}-2x^2yz-2y^4+\frac{2x^2y^3}{z}.$

51. $\frac{a^4}{b^2c^2}+\frac{b^2}{c^2}+\frac{c^2}{b^2}-2\frac{a^2}{b^2}-2\frac{a^2}{c^2}+2.$

52. $x^{2a}-4x^{a+b}+4x^{2b}.$

53. $a^3x^3(ax+2)+a^2x^2(2y+1)+2axy+y^2.$

54. $a^2x^4-2abx^3-(2a-1)\,b^2x^2+2b^3x+b^4.$

55. $x^2\left(1+\frac{1}{n}+\frac{1}{4n^2}\right)-m\left(2+\frac{1}{n}\right)+\frac{m^2}{x^2}.$

56. $\frac{x^2}{y^2}+\frac{y^2}{x^2}-4m\left(\frac{x}{y}+\frac{y}{x}\right)+2\,(2m^2+1).$

57. $x^6+2ax^2-2x+\frac{a^2}{x^2}-\frac{2a}{x^3}+\frac{1}{x^4}.$

58. $\frac{x^2}{y^2}-2\frac{y}{x}+\frac{y^2}{z^2}+2\frac{z}{y}+\frac{z^2}{x^2}-2\frac{x}{z}.$

59. $\frac{x^2}{y^2}-2\frac{x}{y}+\frac{y^2}{x^2}-2\frac{y}{x}+3.$

60. $\left(\frac{x}{y}+\frac{y}{x}\right)^2-4\left(\frac{x}{y}-\frac{y}{x}\right).$

Extract the square root of

61. $a^4-6a^3b+11a^2b^2-6ab^3+b^4.$

62. $1-4a+2a^2+4a^3+a^4.$

63. $a^4+4a^3b+10a^2b^2+12ab^3+9b^4.$

64. $a^4-2a^3+2a^2-a+\frac{1}{4}.$

65. $1+4x+6x^2+4x^3+x^4.$

66. $x^4-6x^3+9x^2-4x+12+\frac{4}{x^2}.$

67. $x^8-4x^7y+4x^6y^2+x^3y^5-2x^2y^6+\frac{y^{10}}{4x^2}.$

68. $x^4+x^2-2x\,(x^2+1)+2+\frac{1}{x^2}.$

69. $x^6+2x^3+1-4x^2\,(x^3-x^2+1).$

70. $a^2x^6-2ax^5+x^4-2x^3+\frac{2ax^2+1}{a^2}.$

71. $529x^4-46x^3+x^2+\frac{289}{x^2}+782x-34.$

72. $1-2a\,(1+a^2-a^4+a^5)+a^2\,(3-a^2+a^6).$

73. $\frac{1}{4a^2} - a^2 + \frac{1}{a} - 3a + 2a^3 + a^4.$

74. $a^4 - 4a^3c + 2a^2c\,(b+2c) - 4abc^2 + b^2c^2.$

75. $\frac{4x^2}{9y^2} - \frac{3y}{x} + \frac{4y^2}{z^2} + \frac{9z^2}{16x^2} + \frac{z}{x} - \frac{8x}{z}.$

Find by inspection the cube root of

76. $27a^3b^3 - 108a^2b^2c^2 + 144abc^4 - 64c^6.$

77. $8x^3 + 12x^2y + 6xy^2 + y^3.$

78. $x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right).$

79. $x^6 - 15x^4 + 75x^2 - 125.$

80. $24389 - 2523a + 87a^2 - a^3.$

81. $x^6 - 6x^5 + 3x^4\,(y+4) - 4x^3\,(3y+2) + 3xy\,(xy + 4x - 2y^2) + y^3.$

82. $x^6 - 3ax^5 + 3\,(a^2+b)\,x^4 - a\,(a^2+6b)\,x^3 + 3a^2bx^2 - 3ab^2x + b^3.$

83. $x^3 - 9xy\,(x-3y) + 3zx\,(z+x) - 9yz\,(3y+z) - 18xyz - 27y^3 + z^3.$

84. $x^3 - \frac{1}{x^3} + 3\left(x^2 + \frac{1}{x^2}\right) - 5.$

85. $\frac{x^3}{y^3} + \frac{y^3}{z^3} + \frac{z^3}{x^3} + 3\left(\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} + \frac{yz}{x^2} + \frac{zx}{y^2} + \frac{xy}{z^2}\right) + 6.$

Extract the cube root of

86. $x^6 - 6x^5y + 9x^4y^2 + 4x^3y^3 - 3x^2y^4 - 12xy^5 - y^6.$

87. $1 - 3x + 5x^3 - 3x^5 - x^6.$

88. $a^9 - a^6 - \frac{1}{a^3} - 3\,(a^8 - a^7 + a^5 - 2a^4 + a^3 - a + 1).$

89. $\frac{x^6}{27} + \frac{x^5}{3} + x^4 - x^3\,(y-1) - 3y\,(2x^2 + 3x - 3y) + 27\,\frac{y^2}{x^2}\left(x - \frac{y}{x}\right).$

90. $1 - x^{12} - 3x\,(1 - x + x^9 - x^{10}) + 2x^3\,(1 - x^6) - 9x^4\,(1 - x + x^3 - x^4).$

Find the fourth root of

91. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4.$

92. $16x^8 - 96a^2x^6 + 216a^4x^4 - 216a^6x^2 + 81a^8.$

93. $256a^4 - 160a^2b\,(8a - 5b) + 25b^2\,(64a^2 - 80ab + 25b^2).$

94. $\frac{a^2}{b^2}\left(\frac{a^2}{b^2} + 4\right) + \frac{b^2}{a^2}\left(\frac{b^2}{a^2} + 4\right) + 6.$

95. $1 + 4x + 6x^2 - 11x^4 - 12x^5 + 2x^6 + 12x^7 + 6x^8 - 4x^9 - 4x^{10} + x^{12}.$

Find the sixth root of

96. $x^6+\frac{1}{x^6}+6\left(x^4+\frac{1}{x^4}\right)+15\left(x^2+\frac{1}{x^2}\right)+20.$

97. $64-96x+60x^2-20x^3+\frac{15}{4}x^2-\frac{3}{8}x^5+\frac{x^6}{64}.$

Find the eighth root of

98. $x^8+\frac{1}{x^8}-8\left(x^6+\frac{1}{x^6}\right)+28\left(x^4+\frac{1}{x^4}\right)-56\left(x^2+\frac{1}{x^2}\right)+70.$

99. $x^{16}+16x^{14}+112x^{12}+448x^{10}+1120x^8+1792x^6+1792x^4+1024x^2+256.$

100. Find the ninth root of

$$x^9-9x^8+36x^7-84x^6+126x^5-126x^4+84x^3-36x^2+9x-1.$$

CHAPTER XI.

SIMPLE EQUATIONS.

90. An **equation** is a statement that two expressions are equal to each other; when the statement is true for all values of the letters involved, the equation is called an identical equation or more usually an **identity.**

An equation is generally only true for particular values of the letters and so is only conditionally true, while an identity is always and unconditionally true. Thus the equation $2x = 10$ is only true if $x = 5$, but the identity $x^2 - 4 = (x - 2)(x + 2)$ is true always and for all values of x.

The letters generally used for the unknown quantities in equations are x, y, z, the letter employed being x if there is only one unknown quantity, and x and y being employed if there are two unknown quantities.

To **solve** an equation means to find the values of the unknown quantity which make it true, and the values so found are called the **roots** of the equation and are said to **satisfy** the equation. The two expressions connected by the sign of equality are called the **sides** of the equation and are distinguished by being on the right or left.

91. A **simple equation** contains the unknown quantity or quantities to the first degree only and so is also called an equation of the first degree; thus $3x = 8$ or $ax = b$ is a simple equation in x, and $2x = 5y$ or $ax - by = c$ is a simple equation in x and y. The degree of the term of highest degree in the unknown quantity or quantities which occurs in an equation is the degree of the equation.

The solution of simple equations depends only on the following four general principles:

If equals be added to equals the sums are equal.

If equals be subtracted from equals the remainders are equal.

If equals be multiplied by equals the products are equal.

If equals be divided by equals the quotients are equal.

92. It follows at once from the first and second of these principles that any term may be transposed from one side of an equation to the other by merely changing its sign. For if

$$a = b - c,$$

adding c to both sides of the equation, we have

$$a + c = b - c + c,$$

i.e.
$$a + c = b;$$

so that the quantity $-c$ has disappeared from the right side and re-appears on the left with its sign changed as $+c$.

Again subtracting a from each side of the last equation we have

$$a + c - a = b - a,$$

i.e.
$$c = b - a;$$

so that the quantity a has disappeared from the left side and re-appears on the right with its sign changed as $-a$.

By the same principles any term which appears on both sides of an equation may be at once cancelled since this is equivalent to adding the same quantity to both sides or subtracting the same quantity from both sides.

Thus
$$a - c = b - c,$$

or
$$a + c = b + c$$

is at once reduced to
$$a = b.$$

In the same way the third principle permits us to change the sign of both sides of an equation simultaneously, for this is the same as to multiply both sides by -1. Thus the equation

$$a = b - c$$

is the same as
$$-a = -b + c;$$

a result which may also at once be obtained by transposing the terms from one side of the equation to the other.

And the fourth principle permits us to remove a common factor which occurs on both sides of an equation.

93. It will be found that a simple equation involving one unknown quantity may generally be solved by the following rule:

Clear the equation of fractions, if any, and remove all brackets. Then transpose all terms containing the unknown quantity to one side and all the other terms to the other side. Finally divide both sides by the coefficient of the unknown quantity and the required root is obtained.

Sometimes however it is advantageous not to clear an equation of fractions at once but to do it gradually or by pairs of terms; and sometimes artifices have to be employed which are illustrated in the typical examples here solved.

Example 1. Solve the equation

$$10x-4=3x+73.$$

Transposing $3x$ to the left and -4 to the right by changing their signs, we have

$$10x-3x=73+4,$$

i.e.
$$7x=77;$$

and dividing both sides by 7,

$$x=11.$$

Example 2. Solve $\dfrac{x}{6}-\dfrac{1}{2}=\dfrac{x}{3}-\dfrac{x}{4}$.

Multiplying both sides by 12, the L. C. M. of the denominators, we have

$$2x-6=4x-3x;$$

and collecting the terms in x to the left and transposing -6 to the right,

$$2x-4x+3x=6;$$

$$\therefore\ x=6.$$

In an equation like this it is convenient, though not necessary, to get rid of fractional coefficients by multiplying both sides by the L. C. M. of all the denominators.

Example 3. Solve

$$7(x-2)-9(1-4x)=6(5x-1)-2(1-8x).$$

Removing all brackets we have

$$7x-14-9+36x=30x-6-2+16x;$$

Transposing,

$$7x+36x-30x-16x=-6-2+14+9,$$

i.e.
$$-3x=15;$$

and dividing both sides by -3,

$$x=-5.$$

Example 4. Solve

$$(2x-3)(3x+4)=6(x-1)^2+4.$$

Multiplying out and squaring we have

$$6x^2-x-12=6x^2-12x+6+4;$$

and cancelling the term $6x^2$ which is common to both sides,

$$-x-12=-12x+6+4.$$

Collecting the terms in x to the left and the numerical terms to the right,

$$12x-x=12+6+4,$$

i.e.
$$11x=22;$$

and dividing both sides by 11,

$$x=2.$$

In this example the equation is originally of a degree apparently higher than the first and so might be supposed not to be a simple equation, but on simplification it is found that the terms of a degree higher than the first cancel each other, and so the equation is reduced to a simple equation.

Example 5. Solve

$$\frac{7x-3}{2x-1}-\frac{5x-2}{4x-3}=\frac{9}{4}.$$

Multiplying by $4(2x-1)(4x-3)$, the L.C.M. of the denominators, we have

$$4(4x-3)(7x-3)-4(2x-1)(5x-2)=9(2x-1)(4x-3);$$

multiplying out and removing all brackets,

$$112x^2-132x+36-40x^2+36x-8=72x^2-90x+27;$$

$$\therefore\ 72x^2+40x^2-112x^2+132x-36x-90x=36-8-27;$$

i.e.
$$6x=1;$$

$$\therefore\ x=\tfrac{1}{6}.$$

In this example, as in the last, the terms in x^2 cancel each other, and the equation is reduced to the ordinary form of a simple equation.

Example 6. Solve

$$\frac{1}{x-2}+\frac{1}{x-3}=\frac{1}{x-1}+\frac{1}{x-4}.$$

The equation may be written in the form

$$\frac{1}{x-2}-\frac{1}{x-1}=\frac{1}{x-4}-\frac{1}{x-3};$$

$$\therefore \quad \frac{1}{(x-1)(x-2)}=\frac{1}{(x-3)(x-4)};$$

$$\therefore \quad (x-1)(x-2)=(x-3)(x-4);$$

$$\therefore \quad x^2-3x+2=x^2-7x+12.$$

Cancelling the common term x^2 on both sides and collecting the terms in x to one side and the numerical terms to the other,

$$4x=10;$$

$$\therefore \quad x=\tfrac{5}{2}.$$

The solution of the equation

$$\frac{x-1}{x-2}+\frac{x-2}{x-3}=\frac{x}{x-1}+\frac{x-3}{x-4},$$

may be immediately deduced from the foregoing, for it may be written in the form

$$\frac{x-2+1}{x-2}+\frac{x-3+1}{x-3}=\frac{x-1+1}{x-1}+\frac{x-4+1}{x-4},$$

i.e.

$$1+\frac{1}{x-2}+1+\frac{1}{x-3}=1+\frac{1}{x-1}+1+\frac{1}{x-4};$$

$$\therefore \quad \frac{1}{x-2}+\frac{1}{x-3}=\frac{1}{x-1}+\frac{1}{x-4}.$$

Or it may be solved independently by arranging it in the form

$$\frac{x-1}{x-2}-\frac{x}{x-1}=\frac{x-3}{x-4}-\frac{x-2}{x-3},$$

so that

$$\frac{1}{(x-1)(x-2)}=\frac{1}{(x-3)(x-4)};$$ as before.

Example 7. Solve

$$\cdot 75x-\frac{\cdot 24x-\cdot 066}{\cdot 3}=4(\cdot 18x+\cdot 895)-2(1\cdot 18-\cdot 115x).$$

Removing the denominator and brackets, we have

$$\cdot 75x-\cdot 8x+\cdot 22=\cdot 72x+3\cdot 58-2\cdot 36+\cdot 23x;$$

$$\therefore \quad (\cdot 72+\cdot 23+\cdot 8-\cdot 75)x=\cdot 22+2\cdot 36-3\cdot 58;$$

i.e.

$$(1\cdot 75-\cdot 75)x=2\cdot 58-3\cdot 58;$$

$$\therefore \quad x=-1.$$

In examples like this which involve decimals it is frequently more convenient to express the decimals at once as vulgar fractions, and this should always be done in the case of circulating decimals.

Example 8. Solve

$$a^2(a-x)-b^2(b+x)=abx.$$

Removing the brackets,

$$a^3-a^2x-b^3-b^2x=abx;$$

$$\therefore\ (a^2+ab+b^2)\,x=a^3-b^3;$$

$$\therefore\ x=a-b.$$

Example 9. Solve

$$\sqrt{x}+\sqrt{x+4a}=2\sqrt{x-b}.$$

Squaring, we have

$$2x+4a+2\sqrt{x^2+4ax}=4x-4b;$$

$$\therefore\ \sqrt{x^2+4ax}=x-2(a+b);$$

squaring again, $x^2+4ax=x^2-4(a+b)\,x+4(a+b)^2;$

cancelling the common term x^2 on both sides and then dividing by 4,

$$ax=-(a+b)\,x+(a+b)^2;$$

$$\therefore\ (2a+b)\,x=(a+b)^2;$$

$$\therefore\ x=\frac{(a+b)^2}{2a+b}.$$

In examples of this class the radical signs must gradually be removed by squaring and re-arranging the terms so that a single radical term may be on one side by itself.

SIMULTANEOUS SIMPLE EQUATIONS.

94. Two or more equations involving the same number of unknown quantities and satisfied by the same values of the unknown quantities are called **simultaneous** equations; and if the equations be of the first degree in the unknown quantities they are called simultaneous simple equations.

It is readily seen that a single equation containing two unknown quantities has an infinite number of solutions. For example the equation

$$3x-4y=12$$

is satisfied by $x=4$, $y=0$, or by $x=0$, $y=-3$, or by $x=8$, $y=3$, and so on; and generally if we assign any value we please to x the corresponding value of y is given by

$$y=\frac{3x-12}{4},$$

so that there is no limit to the number of solutions.

But if we have a second equation connecting x and y and which is to be satisfied by the same values of x and y as the first, such as

$$x-y=5;$$

then from this second equation,

$$y=x-5,$$

so that $$\frac{3x-12}{4}=y=x-5.$$

Hence, multiplying by 4,

$$3x-12=4x-20;$$

$$\therefore\ 4x-3x=20-12;$$

$$\therefore\ x=8;$$

$$\therefore\ y=x-5=8-5=3.$$

Thus the only possible solution is given by $x=8$, $y=3$.

The two equations presented for solution must be *independent*, i.e. not deducible from each other; for otherwise we have really only one equation to determine two unknown quantities, and the number of solutions will be infinite. Thus the equations

$$3x-4y=12,$$

$$8y+24=6x,$$

are not independent, for the second is at once deduced from the first by multiplying by 2 and arranging the terms in a different order. Each equation thus expresses the same relation between x and y, and we have only one equation involving two unknown quantities and so having an infinite number of solutions. But the equations

$$3x-4y=12,$$

$$x-\ y=\ 5,$$

are independent and yield only the result $x=8$, $y=3$.

95. One method of solving two simultaneous simple equations has been already suggested, viz. to solve each of them for the same unknown quantity and then to equate to each other the two values thus obtained; this produces a simple equation containing the other unknown quantity only, which can be solved by methods already explained.

Another method is to multiply the equations by such factors as will make the coefficients of one of the unknown quantities the same in the two equations; then by addition or subtraction an equation is formed which contains the other unknown quantity only.

Or we may solve one of the equations for one of the unknown quantities and substitute the value thus obtained in the other equation.

Any of these three methods enables us to effect the solution of the given equations, and the particular method to be adopted depends on the shape of the equations; the second method is however of most general utility.

The process by which we remove one of the unknown quantities from the given equations is called **elimination.**

Example 1. Solve the simultaneous equations

$$7x-5y=4 \quad \ldots\ldots(1),$$

$$3x+2y=39 \quad \ldots\ldots(2).$$

From (1), $$y=\frac{7x-4}{5},$$

and from (2), $$y=\frac{39-3x}{2};$$

equating these values of y,

$$\frac{7x-4}{5}=\frac{39-3x}{2};$$

multiplying by 10, $$14x-8=195-15x;$$

$$\therefore\ 14x+15x=195+8;$$

i.e. $$29x=203;$$

$$\therefore\ x=7.$$

Then $$y=\frac{7x-4}{5}=\frac{49-4}{5}=\frac{45}{5}=9.$$

Thus the solution required is $x=7$, $y=9$.

Example 2. Solve $13x-4y=10$ .....(1),

$3x+5y=103$ .....(2).

To eliminate y multiply (1) by 5 and (2) by 4; thus

$$65x-20y=50,$$

$$12x+20y=412;$$

adding, $$77x=462;$$

$$\therefore\ x=6.$$

Substituting this value of x in either of the given equations we obtain y. Thus from (2)

$$18+5y=103;$$

$$\therefore\ 5y=85;$$

$$\therefore\ y=17.$$

Example 3. Solve

$$\frac{1}{5}(x-2y)+\frac{1}{6}(3y-2x)=\frac{y}{15}-\frac{x}{10}-\frac{7}{30},$$

$$\frac{2}{3}(2x-5y)-\frac{1}{2}(x-4y)=\frac{y}{6}+\frac{x}{2}.$$

Multiplying the first equation by 30 and the second by 6, getting rid of brackets and collecting like terms, we have

$$x-y=7.....(1),$$

$$2x-9y=0.....(2).$$

From (1), $y=x-7$; substituting in (2),

$$2x-9(x-7)=0;$$

$$\therefore\ 7x=63;$$

$$\therefore\ x=9.$$

Then $$y=x-7=9-7=2.$$

Example 4. Solve $\frac{9}{x}-\frac{4}{y}=6$ .....(1),

$\frac{5}{x}-\frac{3}{y}=1$ .....(2).

To eliminate y multiply (1) by 3 and (2) by 4; thus

$$\frac{27}{x}-\frac{12}{y}=18,$$

$$\frac{20}{x}-\frac{12}{y}=4;$$

subtracting, $$\frac{7}{x}=14;$$

$$\therefore\ \frac{1}{x}=2;$$

$$\therefore\ x=\tfrac{1}{2}.$$

Then $$\frac{3}{y}=\frac{5}{x}-1=10-1=9;$$

$$\therefore\ \frac{1}{y}=3;$$

$$\therefore\ y=\tfrac{1}{3}.$$

In equations of this shape the unknown quantities should be taken to be $\frac{1}{x}, \frac{1}{y}$ and not x, y; and the equations should not be cleared of fractions other than numerical before commencing the solution, as this would produce equations of the second degree in x, y.

Example 5. Solve

$$xy-3x-2=(x+2)(y-3)=1.$$

First, getting rid of brackets,

$$xy-3x-2=xy-3x+2y-6;$$

cancelling common terms on both sides and collecting the numerical terms on one side,

$$2y=4;$$

$$\therefore\ y=2.$$

Then $$1=xy-3x-2=2x-3x-2;$$

$$\therefore\ x=-3.$$

The equation here given is apparently of a degree higher than the first, but on simplification is readily reduced to a simple equation by the terms of a higher degree cancelling each other.

Example 6. Solve

$$\sqrt{x+y}-\sqrt{y}=\sqrt{x+2y-4a},\quad \sqrt{x+y-3a}+\sqrt{y}=\sqrt{x+2y-a}.$$

Squaring the first equation,

$$x+2y-2\sqrt{xy+y^2}=x+2y-4a;$$

$$\therefore\ \sqrt{xy+y^2}=2a;$$

$$\therefore\ xy+y^2=4a^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Squaring the second equation,

$$x+2y-3a+2\sqrt{xy+y^2-3ay}=x+2y-a;$$

$$\therefore\ \sqrt{xy+y^2-3ay}=a;$$

$$\therefore\ xy+y^2-3ay=a^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Then from (1) and (2),

$$3ay+a^2=xy+y^2=4a^2;$$
$$\therefore\ 3ay=3a^2;$$
$$\therefore\ y=a.$$

Also
$$4a^2=xy+y^2=ax+a^2;$$
$$\therefore\ ax=3a^2;$$
$$\therefore\ x=3a.$$

In this class of examples the radicals must gradually be removed by squaring.

96. We have proved that two simultaneous equations are required in order to determine two unknown quantities, and similarly three simultaneous equations are required in order to determine three unknown quantities.

When three such simultaneous equations are given, from any pair of them we can eliminate one of the unknown quantities and from any other pair we can eliminate the same unknown quantity; we thus obtain two equations involving two unknown quantities which can be solved by methods already explained.

As remarked already in the case of two simultaneous equations, the three simultaneous equations given for the determination of three unknown quantities must be *independent*, i.e. any one of them must not be deducible from one of the others or from both.

Example 1. Solve the simultaneous equations

$$3x+5y+2z=1 \quad \ldots\ldots(1),$$
$$2x-7y-4z=16 \quad \ldots\ldots(2),$$
$$5x+2y-9z=2 \quad \ldots\ldots(3).$$

Multiplying (1) by 2, $6x+10y+4z=2,$

and adding this to (2), $8x+3y=18 \quad \ldots\ldots(4).$

Multiplying (1) by 9 and (3) by 2,

$$27x+45y+18z=9,$$
$$10x+4y-18z=4;$$

adding these equations, $37x+49y=13 \quad \ldots\ldots(5).$

From (4) and (5), $\dfrac{18-8x}{3}=y=\dfrac{13-37x}{49};$

$$\therefore\ 882-392x=39-111x;$$
$$\therefore\ 281x=843;$$
$$\therefore\ x=3.$$

Then from (4), $y=\frac{18-24}{3}=-\frac{6}{3}=-2;$

and from (1) $9-10+2z=1;$

$\therefore \quad 2z=2;$

$\therefore \quad z=1.$

Example 2. Solve $\frac{4a}{x}-\frac{3b}{y}-\frac{2c}{z}=3,$

$\frac{b}{y}+\frac{c}{8z}-\frac{a}{4x}=\frac{1}{12},$

$\frac{4c}{3z}+\frac{2a}{3x}+\frac{b}{2y}=0.$

The shape of the equations indicates that it will be convenient to take the unknown quantities to be $\frac{a}{x}, \frac{b}{y}, \frac{c}{z}$ instead of x, y, z, and it will facilitate the solution if we replace $\frac{a}{x}, \frac{b}{y}, \frac{c}{z}$ by x', y', z'.

Making this substitution and multiplying the second equation by 24 and the third by 6 so as to make the numerical coefficients integers, we have for our three equations

$4x'-3y'-2z'=3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$

$-6x'+24y'+3z'=2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$

$4x'+3y'+8z'=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$

Multiplying (1) by 3 and (2) by 2,

$12x'-9y'-6z'=9,$

$-12x'+48y'+6z'=4;$

adding, $39y'=13;$

$\therefore \quad y'=\frac{1}{3}.$

Subtracting (1) from (3), $6y'+10z'=-3;$

$\therefore \quad 2+10z'=-3;$

$\therefore \quad 10z'=-5;$

$\therefore \quad z'=-\frac{1}{2}.$

Adding (1) and (3), $8x'+6z'=3;$

$\therefore \quad 8x'-3=3;$

$\therefore \quad 8x'=6;$

$\therefore \quad x'=\frac{3}{4}.$

Hence $\frac{a}{x}=\frac{3}{4}, \frac{b}{y}=\frac{1}{3}, \frac{c}{z}=-\frac{1}{2}$, so that $x=\frac{4}{3}a, y=3b, z=-2c.$

Example 3. Solve

$$\frac{1}{a}(y+z-x)=\frac{1}{b}(z+x-y)=\frac{1}{c}(x+y-z),$$

$$x+y+z=a+b+c.$$

Let the value of each of the first three equal quantities be k, so that

$$y+z-x=ka,$$

$$z+x-y=kb,$$

$$x+y-z=kc;$$

adding, $$x+y+z=k(a+b+c);$$

but $$x+y+z=a+b+c;$$

$$\therefore\quad k=1;$$

hence $$y+z-x=a,\ z+x-y=b,\ x+y-z=c.$$

Then we have $$x+y+z=a+b+c,$$

$$y+z-x=a;$$

subtracting, $$2x=b+c;$$

$$\therefore\quad x=\tfrac{1}{2}(b+c);$$

similarly $$y=\tfrac{1}{2}(c+a),\ z=\tfrac{1}{2}(a+b).$$

The artifice here illustrated is of frequent use in the solution of equations of this shape; when several quantities are equal to each other, it is of service to represent each by the same symbol k. Useful information may also often be obtained by equating each of several equal quantities to their sum divided by the number of the quantities; thus if

$$y-z=z-x=x-y;$$

each of these three equal quantities

$$=\tfrac{1}{3}(y-z+z-x+x-y)$$

$$=0,$$

so that x, y, z are all equal.

97. In the same manner if we have four simultaneous equations involving four unknown quantities we must eliminate one of the unknown quantities between three pairs of the given equations, thus obtaining three new equations involving three unknown quantities, which can be solved by the methods just explained. And the process is similar for any number of equations involving the same number of unknown quantities.

EXAMPLES.

Solve the following equations:

1. $7x+1=36$.
2. $4x-3=5$.
3. $9-8x=57$.
4. $2x+9=5$.
5. $3x=10-2x$.
6. $24=3+7x$.
7. $5x=16+7x$.
8. $40=14x+47$.
9. $8x+7=5x+13$.
10. $9x-3=7x+11$.
11. $7x+3=13x-1$.
12. $2x-5=3+10x$.
13. $27x-2=22x+4$.
14. $70-8x=10-3x$.
15. $7x=37+2x-12$.
16. $67=43-4x-60$.
17. $419x-11077=272x-9901$.
18. $13x-7-20x+31=17x$.
19. $11x+13-18x-37=5x-24$.
20. $7x-12-23x+41=15x-37-4x-15$.
21. $23-4x-149-11x=100-x-36+5x$.
22. $x-\frac{x}{2}=\frac{x}{3}-\frac{1}{12}$.
23. $\frac{x}{5}-\frac{x}{8}=\frac{x}{10}-1$.
24. $\frac{x}{3}-\frac{2}{5}=\frac{x}{6}+2\frac{3}{5}$.
25. $\frac{x}{2}-\frac{x}{12}=3-\frac{x}{3}$.
26. $\frac{x}{4}-\frac{x}{8}-\frac{x}{16}=2$.
27. $\frac{x}{19}-\frac{x}{38}+\frac{x}{57}=\frac{5}{6}$.
28. $4(x-9)=9(x-4)$.
29. $7(x-3)=11(x+5)$.
30. $4(x-3)-5(10-x)=7(x+5)-9(2x-7)$.
31. $7(2x+5)+4(3x-11)=9(8x-3)+8(3x+4)$.
32. $33-4(2x-3)-11(3x-5)=42-7(3x-4)-10(x-4)$.
33. $15(8-9x)-13(5-2x)+100x=17(2+3x)-19(1-4x)+152x$.
34. $1-[x-6-\{2x-(3x-7)\}]=0$.
35. $32-5x-[8-12x-\{6-7x-(10-3x)\}]=20$.
36. $8x-[3-\{5x-(2-7x)\}]=8-[7-\{4x-(5-6x)\}]$.
37. $\frac{x-5}{3}-\frac{x-3}{7}=\frac{x-1}{4}-\frac{x+1}{9}$.
38. $\frac{2}{3}(x-4)+\frac{3}{4}(x-3)=\frac{5}{6}(x-1)$.
39. $\frac{1}{6}(5x-2)-\frac{1}{8}(3x+2)=\frac{1}{12}(7x+2)-\frac{1}{4}(x-2)$.
40. $\frac{8+17x}{9}-\frac{11-25x}{36}=\frac{9-11x}{4}-\frac{17-19x}{18}-5$.

41. $\dfrac{x-\frac{1}{3}}{6}-\dfrac{\frac{7x}{4}-\frac{4}{3}}{8}=\dfrac{\frac{x}{4}-\frac{1}{6}}{4}$.

42. $\frac{1}{3}\left(\frac{1}{5}-\frac{x}{8}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{x}{4}\right)=\frac{1}{5}\left(\frac{5x}{4}-\frac{1}{2}\right)$.

43. $\frac{4}{7}+\frac{x}{5}-\frac{1}{3}\left(\frac{x}{2}-\frac{7x}{10}\right)=\frac{1}{21}-\frac{1}{11}\left(\frac{7x+10}{9}-1\tfrac{4}{7}\right)$.

44. $(x-2)(x-3)=(x-1)(x-6)$.

45. $(x+4)(2x-1)=2(x+1)^2$.

46. $(4x-3)(7x+6)=(2x-3)(14x+5)$.

47. $(8x-5)(2x+1)=(4x+3)^2$.

48. $12x+(3x-1)(4x+3)=1+(x+2)(12x-1)$.

49. $2x^2=(x-3)^2+(x+5)^2$.

50. $(x-1)(x-2)+(x+2)(x-3)+(x+3)(x-4)=3(x+4)(x-5)$.

51. $25(x^2-1)=(2x-1)(2x+3)+(3x-2)(6x+1)+(x+4)(3x-5)$.

52. $(3x+13)^2-2(x+3)^2=3(x+7)^2+4(x-5)^2$.

53. $(x+2)^3+(x+3)^3+(x+4)^3=3(x+2)(x+3)(x+4)$.

54. $\dfrac{9x+1}{3x+1}-\dfrac{10x+3}{5x+3}=1$.

55. $\dfrac{3x-2}{4x-3}-\dfrac{2x+1}{3x+2}=\dfrac{1}{12}$.

56. $\dfrac{(7x+3)(3x-5)}{(9x+1)(x-2)}=\dfrac{7}{3}$.

57. $\dfrac{4}{(2x+1)(4x-1)}=\dfrac{5}{(2x-1)(5x+1)}$.

58. $\dfrac{2(6x+1)}{11x-8}+\dfrac{5x-4}{9}=\dfrac{10x+17}{18}$.

59. $\dfrac{1}{2}+\dfrac{x+19}{x+12}=\dfrac{3x+4}{2x+1}$.

60. $\dfrac{4x-1}{4x^2+x-1}=\dfrac{6x-1}{6x^2+2x-1}$.

61. $\dfrac{1+x+x^2}{1+x}-\dfrac{1-x+x^2}{1-x}=2x$.

62. $\dfrac{x^2+x}{x^2+x+1}-\dfrac{x^2-x}{x^2-x+1}=\dfrac{2}{x^3+x+1}$.

63. $\dfrac{13-x}{13+x}+\dfrac{8x^2}{169-x^2}=\dfrac{13+x}{13-x}-8$.

64. $\dfrac{1}{x-1}+\dfrac{1}{x-7}=\dfrac{1}{x-3}+\dfrac{1}{x-5}$.

65. $\dfrac{1}{x-2}+\dfrac{1}{x-5}=\dfrac{1}{x-3}+\dfrac{1}{x-4}$.

66. $\dfrac{1}{x}+\dfrac{1}{x+12}=\dfrac{1}{x+4}+\dfrac{1}{x+8}$.

67. $\dfrac{x}{x-1}+\dfrac{x-2}{x-3}=\dfrac{x+1}{x}+\dfrac{x-3}{x-4}$.

68. $\frac{x-1}{x}+\frac{x-2}{x-1}=\frac{x+1}{x+2}+\frac{x-4}{x-3}$.

69. $\frac{x-4}{x-5}+\frac{x+7}{x+6}=\frac{x-1}{x-2}+\frac{x+4}{x+3}$.

70. $\frac{7x-132}{x-19}-\frac{6x-125}{x-21}=\frac{9x-98}{x-11}-\frac{8x-103}{x-13}$.

71. $\frac{6x+93}{x+15}+\frac{8x+43}{x+5}=\frac{3x+54}{x+17}+\frac{11x+36}{x+3}$.

72. $\frac{10x-97}{x-9}-\frac{4x-99}{x-23}=\frac{5x-142}{x-27}+\frac{x-34}{x-41}$.

73. $\cdot734x-4\cdot678=\cdot347x+1\cdot822-\cdot113x$.

74. $\frac{\cdot45x}{1\cdot5}-\frac{\cdot255-\cdot204x}{\cdot12}=\frac{\cdot0703}{\cdot37}+\frac{\cdot016x-\cdot0119}{\cdot004}$.

75. $1\cdot7\times(\cdot37x+\cdot47)-1\cdot9\times(\cdot51x-\cdot71)=4\cdot7\times(\cdot23-\cdot15x)+1\cdot432$.

76. $\cdot\dot1\dot2x-\cdot5\dot7\dot5+\cdot2\dot1\dot2x=\cdot8\dot3x+\cdot4\dot2\dot4-\cdot\dot6x$.

77. $\cdot1\dot8\dot7+\cdot\dot3x-\cdot02\dot7=1\cdot3\dot4\dot3+\cdot2\dot7x-1\cdot1\dot5$.

78. $2(x-a)+5(x-3a)=8(x-5a)$.

79. $a(x-a)-b(x-b)=0$.

80. $ab(x-ab)=a^2(a^2-x)+b^2(b^2-x)$.

81. $a^2(x-a)+b^2(x-b)=abx$.

82. $a(x+b)-4b(x-8b)=(a+2b)^2$.

83. $a(a+b-x)=3b(x+2b)$.

84. $a(a-x)+5ab=9b(4b+x)$.

85. $3b(3b-x)-5a(4a-x)=3ab$.

86. $\frac{1}{3}(x+a)-\frac{1}{4}(3x+5a)+\frac{1}{6}(7x+13a)=\frac{1}{8}(9x-5a)$.

87. $\frac{a}{bx}-\frac{b}{ax}=1-\frac{b^2}{a^2}$.

88. $\frac{a}{b}\left(x-\frac{b}{a}\right)+\frac{b}{a}\left(x-\frac{a}{b}\right)=\frac{(a-b)^2}{ab}$.

89. $\frac{3a-5x}{4a}+\frac{2b-7x}{8b}=1-\frac{9x}{a+b}$.

90. $(x+a)(x+b)-(x-a)(x-b)=2(a+b)$.

91. $(x-a^2)(x-b^2)=(x-ab)^2$.

92. $2(x+a)(x+b)=x(x+4a)+x(x+4b)$.

93. $(ax+b)(bx-a)=(abx-1)(x+ab)$.

94. $(a+b)(x-a+b)^2+(a-b)(x-a-b)^2=2ax^2$.

95. $(x-a)^2+(x-b)^2=(a-b)^2+2x^2$.

96. $(x-a+b)(x+a-b)=(x-a-b)^2$.

97. $a(x^2+x-a^2)+b(x^2+x-b^2)=(a+b)(x^2+3ab)$.

98. $(x-b)(x+c)+(x-c)(x+a)+(x-a)(x+b)=x(3x-1)$.

99. $a(x-a)^2+b(x-b)^2+c(x-c)^2$
$=ax(x-2b)+bx(x-2c)+cx(x-2a)+3abc.$

100. $(x-bc)(x-ca)=x(x+ab)-bc(ab+bc).$

101. $(a-x)^3+(b-x)^3=\frac{1}{4}(a+b-2x)^3.$

102. $\frac{x+a}{x-a}+\frac{x+b}{x-b}=2.$

103. $\frac{x-a+b}{x-a-b}=\frac{x+a-b}{x+a+b}.$

104. $\frac{b}{bx-a}-\frac{a}{ax-b}=\frac{a^2-b^2}{abx^2}.$

105. $\frac{x^2}{x-a}-\frac{x^2}{x-b}=a-b.$

106. $\frac{1}{x-a-b}+\frac{1}{x-c}=\frac{1}{x-a-c}+\frac{1}{x-b}.$

107. $\frac{2}{x-a}+\frac{1}{x-2b}=\frac{2}{x-b}+\frac{1}{x-2a}.$

108. $\frac{1}{\frac{x}{2}-a-1}-\frac{1}{\frac{x}{2}-a}=\frac{1}{\frac{x}{2}-b-1}-\frac{1}{\frac{x}{2}-b}.$

109. $\frac{x-a}{x-a-1}+\frac{x-b+1}{x-b}=\frac{x-b}{x-b-1}+\frac{x-a+1}{x-a}.$

110. $\frac{1}{x-a+b}+\frac{1}{x+a+c}=\frac{1}{x-a+c}+\frac{1}{x+a+b}.$

111. $\frac{2b-x}{(a+b-x)^2}=\frac{2c-x}{(a+c-x)^2}.$

112. $\sqrt{x+7}-\sqrt{x-5}=2.$

113. $\sqrt{x-\sqrt{x-144}}=8.$

114. $\sqrt{63+5x}+\sqrt{48+5x}=15.$

115. $\sqrt{x+1}+\sqrt{x-1}=\frac{2}{\sqrt{x+1}}.$

116. $x+\sqrt{a^2+x^2}=a.$

117. $a-\sqrt{2ax+x^2}=x.$

118. $\sqrt{x+a^2}-\sqrt{x+b^2}=a-b.$

119. $\frac{\sqrt{x+a}-\sqrt{x-a}}{\sqrt{x+a}+\sqrt{x-a}}=1.$

120. $\sqrt{x-a}+\sqrt{x+a}=2\sqrt{a}.$

Solve the simultaneous equations:

121. $3x-2y=7,\ 2x+3y=22.$

122. $17x-4y=9,\ 5x-y=15.$

123. $11x+4y=5,\ 7x+3y=0.$

124. $40x=4-3y,\ 8x=12+5y.$

125. $2=3x+2y,\ \frac{1}{6}=5y-7x.$

126. $\frac{1}{6}x=6-\frac{1}{4}y,\ \frac{1}{6}y=5\frac{2}{3}-\frac{1}{4}x.$

127. $\frac{2x}{5}-\frac{y}{3}=\frac{1}{6},\ \frac{9x}{10}-2y=\frac{1}{2}.$

128. $\frac{x}{8}+\frac{y}{7}-8=0,\ \frac{11x}{12}-\frac{3y}{5}-1=0.$

129. $\frac{13x-2y}{15}-\frac{10x-31y}{9}=7=\frac{5x+2y}{11}+\frac{x-2y}{6}.$

130. $\frac{x}{3}-\frac{y}{5}=0=\frac{x}{5}-\frac{y}{3}$.

131. $24x=8x-6y+5=20x+12y-1$.

132. $5(8x-9y)=11(14x-11y)$, $7(3x+8)=2(5y+22)$.

133. $\frac{x+3}{4}=\frac{y+11}{3}=\frac{x-y}{5}$.

134. $\frac{40-x}{11}=\frac{40-y}{13}=\frac{x-y}{2}$.

135. $5(x-y)=4(x+y)=6x-7$.

136. $3(4x-3y-1)=5(x+y-2)=7(5x-2y-3)$.

137. $1-\frac{1-3y}{11}=2-\frac{5-2x}{7}=3-\frac{x-3y}{16}$.

138. $\frac{3}{3(x+y)-7}+\frac{3}{3(x-y)-7}=\frac{1}{3(x+y)-7}-\frac{1}{3(x-y)-7}=\frac{3}{4}$.

139. $\cdot 3x-\cdot 7y=\cdot 05$, $\cdot 9x+\cdot 14y=\cdot 31$.

140. $\cdot 25x+4\cdot 25-\cdot 225y=\cdot 75x$, $\cdot 93x+3\cdot 65-\cdot 29y=4\cdot 47$.

141. $\frac{3}{x}+\frac{4}{y}=11$, $\frac{7}{x}-\frac{3}{y}=1$.

142. $\frac{5}{x}-\frac{4}{y}=2$, $\frac{9}{x}-\frac{8}{y}=6$.

143. $\frac{3}{x}-\frac{2}{y}=\frac{1}{2}$, $\frac{15}{y}-\frac{11}{x}=\frac{1}{12}$.

144. $\frac{4}{x}-\frac{5}{y}+2=0$, $\frac{5}{x}-\frac{6}{y}=0$.

145. $\frac{1}{4x}+\frac{4}{y}=0=\frac{1}{x}-\frac{8}{y}-6$.

146. $x+\frac{3}{y}=\frac{7}{6}$, $\frac{4}{y}-\frac{x}{2}=\frac{5}{4}$.

147. $\frac{4}{x}-3y=2=\frac{10}{x}-8y$.

148. $xy+x+1=(x-1)(y+1)=2$.

149. $xy-2y+1=(x-2)(y+1)=0$.

150. $(x-1)(y+2)=(x-3)(y+3)$, $(2x-9)(y+19)=2(x-2)(y+4)$.

151. $(4x+1)(y-1)=(2x-3)(2y+1)=(x+4)(4y-9)$.

152. $(x+1)(y-4)=(x-3)(y+3)=(x-5)(y+2)$.

153. $x+y=a+b$, $ax+by=2ab$.

154. $ax+by=a^2+b^2$, $ax-by=a^2-b^2$.

155. $a^2x+b^2y=a^3+b^3$, $bx=ay$.

156. $(a+b)x+(a-b)y=(a+b)^2$, $(a-b)x+(a+b)y=a^2-b^2$.

157. $(a-b)x+(a+b)y=2(a^2+b^2)$, $(a+b)x=(a-b)y$.

158. $(a+b)x+(a-b)y=a^2+b^2$, $(a-b)x+(a+b)y=a^2-b^2$.

159. $(a-b)x-(a+b)y=2b(a^2-b^2)$, $(a+b)x+(a-b)y=2a(a^2+b^2)$.

160. $2a^3+ab(x-y)=(a^2+b^2)(x+y)$, $2b^3+ab(x+y)=(a^2+b^2)(x-y)$.

161. $ab(x+y)=(a^2+b^2)(x-y)$, $2(x-y)=x+y-2(a-b)^2$.

162. $\frac{x}{a}-\frac{y}{b}=0$, $\frac{x}{b}-\frac{y}{a}=0$.

163. $\frac{x}{a}=\frac{y}{b}=2-\frac{x}{a}$.

164. $\frac{x}{b}-\frac{y}{a}=\frac{a}{b}$, $\frac{y}{b}-\frac{x}{a}=\frac{b}{a}$.

165. $\frac{a-b}{x}+\frac{a+b}{y}=2$, $(a+b)x=(a-b)y$.

166. $\frac{a}{x}-\frac{a}{y}=b$, $\frac{b}{x}+\frac{b}{y}=a$.

167. $\frac{a}{x}-\frac{a}{y}=2=\frac{b}{x}+\frac{b}{y}$.

168. $(a-b)\left(\frac{a^2}{x}+\frac{b^2}{y}\right)=\frac{a^3}{x}-\frac{b^3}{y}=a^2-b^2$.

169. $\sqrt{x-a}+\sqrt{y}=\sqrt{x+y+a}$, $\sqrt{x+2a}-\sqrt{y}=\sqrt{x+y-2a}$.

170. $\sqrt{x}-\sqrt{x-y}=\sqrt{a-y}$, $2\sqrt{x-y}=3\sqrt{a-y}$.

Solve the simultaneous equations:

171. $3x+2y+z=10$, $2x-7y+6z=6$, $9x+4y-5z=2$.

172. $2x+11y-3z=5$, $3x+5y+2z=11$, $x-2y+z=10$.

173. $3x+4y+2z=3$, $8y+12z-21x=1$, $10z-9x+12y=5$.

174. $x+4y-3z=0$, $8x-9y+7z=5x-13y+10z=1$.

175. $x+y+z=2x-y+3z=5x+13y-z=1$.

176. $x-y-z=0$, $x-3y+3z=0$, $x+y+z-1=0$.

177. $8x-9y+2z=\frac{1}{4}(5x+11y-12z)=\frac{1}{6}(2x+y+3z)=2$.

178. $7x=65-y-5z$, $2y=8+4z-3x$, $3z=33-5x+3y$.

179. $x+\frac{y}{2}-z=0$, $\frac{x}{3}-y+\frac{z}{2}=\frac{1}{4}$, $x-\frac{y}{4}-\frac{z}{8}=\frac{1}{2}$.

180. $\frac{x}{6}+\frac{y}{2}-\frac{z}{2}=1$, $\frac{x}{4}-\frac{y}{3}+\frac{z}{8}=2$, $\frac{x}{2}+\frac{y}{6}-\frac{z}{4}=5$.

181. $\frac{x}{2}+\frac{y}{2}+\frac{9z}{8}=2\frac{1}{12}$, $\frac{3x}{5}+\frac{2y}{3}+\frac{z}{4}=3\frac{1}{9}$, $\frac{7x}{10}-\frac{y}{3}+\frac{z}{2}=2\frac{1}{18}$.

182. $\frac{x}{4}+y+2z=\frac{443}{72}$, $\frac{5x}{6}-y+4z=\frac{433}{36}$, $2x-\frac{5y}{3}+z=\frac{85}{27}$.

183. $\frac{y-8z}{5-2x}=\frac{12z-x}{6+y}=\frac{3x-5y}{1+7z}=4$.

184. $x+\frac{1}{10}(y-z)=21$, $y+\frac{1}{11}(z-x)=51$, $z+\frac{1}{12}(x-y)=65$.

185. $\frac{1}{10}(x+y)-\frac{1}{7}(y+z)+\frac{1}{2}(z+x)=\frac{4}{5}(y-x)+\frac{2}{3}(z-y)+\frac{3}{4}(x-z)$
$=\frac{3}{10}(x-y)+\frac{1}{3}(y-z)+\frac{1}{2}(z-x)=1$.

186. $\frac{4}{y}-\frac{3}{z}=\frac{6}{z}-\frac{1}{x}=\frac{6}{x}-\frac{10}{y}=1$.

187. $\frac{4}{x}-\frac{5}{y}-\frac{3}{z}=2$, $\frac{2}{x}+\frac{7}{y}+\frac{6}{z}=3$, $\frac{9}{x}-\frac{10}{y}-\frac{8}{z}=7$.

188. $3\left(\frac{1}{z}-\frac{1}{y}\right)+\frac{1}{x}=\frac{1}{6}$, $3\left(\frac{1}{x}-\frac{1}{z}\right)-\frac{1}{y}=\frac{1}{2}$, $\frac{1}{x}+\frac{1}{y}-\frac{3}{z}=\frac{1}{6}$.

189. $\frac{1}{x}-\frac{1}{y}-\frac{1}{2z}=0$, $\frac{1}{4x}+\frac{1}{3}\left(\frac{1}{y}+\frac{1}{z}\right)=\frac{2}{9}$, $\frac{1}{3y}+\frac{1}{2}\left(\frac{1}{z}+\frac{1}{x}\right)=\frac{1}{3}$.

190. $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=3\left(\frac{c}{z}-\frac{a}{x}\right)=6\left(\frac{b}{y}-\frac{a}{x}\right)=6$.

191. $\frac{x}{a}+\frac{y}{b}-\frac{z}{c}=0$, $\frac{y}{b}+\frac{z}{c}-\frac{x}{a}=4$, $\frac{z}{c}+\frac{x}{a}-\frac{y}{b}=2$.

192. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=4\frac{x}{a}-5\frac{y}{b}+9\frac{z}{c}=7\frac{x}{a}-11\frac{y}{b}-12\frac{z}{c}=3$.

193. $ax+by+cz=(b+c)\,x+(c+a)\,y+(a+b)\,z=0$,
$bcx+cay+abz=abc$.

194. $a\,(y+z)=b\,(z+x)=c\,(x+y)=2$.

195. $y-z=z-x=x-y$, $x+y+z=3a$.

196. $(b-c)\,x+(c-a)\,y+(a-b)\,z=0$, $y-z+3\,(b-c)=0$, $x+y+z=0$.

197. $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, $x+y-z=a^2+b^2-c^2+2ab$.

198. $x+\frac{a}{3}=y+\frac{b}{3}=z+\frac{c}{3}$, $3\,(x+y+z)=2\,(a+b+c)$.

199. $x\,(a-y)=y\,(b-x)$, $z\,(c+x)=x\,(a+z)$, $ax+by=ab$.

200. $x+y+z+u=3x-2y+z-3u=2x+y-z-4u=5$,
$\frac{x}{2}-\frac{y}{3}-\frac{z}{9}-\frac{u}{2}=0$.

CHAPTER XII.

PROBLEMS.

Problems which lead to Simple Equations.

98. A problem which produces an equation or equations is a statement of facts, real or supposed, in ordinary life, which has to be translated into algebraical language; and the symbolical expression of the conditions of the problem constitutes the only difficulty in dealing with this class of questions.

In any problem presented for solution we must substitute one or more of the symbols x, y, z for the unknown quantities which occur; we then proceed with these quantities x, y, z as if they were already known, and make them satisfy all the given conditions, thus forming as many independent equations as there are unknown quantities; and finally the solution of these equations determines the unknown quantities.

99. In the solution of a problem it is frequently in our power to select either one unknown quantity or two unknown quantities and so to form either one equation involving one unknown quantity or two simultaneous equations involving two unknown quantities; and similarly we may often for the solution of the same problem form two simultaneous equations involving two unknown quantities or three simultaneous equations involving three unknown quantities. We must be guided by circumstances and the nature of the problem in deciding how many unknown quantities to have, and therefore how many symbols to employ in the solution of a particular problem; but the general rule may be laid down that the smaller the number of unknown quantities consistent with clearness of statement the better. Sometimes however considerations of symmetry

render the solution of a problem easier by using two unknown quantities than by using only one; and similarly, when three quantities are symmetrically involved, it is often more advantageous to employ three symbols than two.

100. *Problems which lead to equations involving one unknown quantity.*

Example 1. Find two numbers such that their difference is 48 and the greater is seven times the less.

Let x be the smaller number, then $x+48$ is the greater and is seven times the less. Hence

$$x+48=7x;$$
$$\therefore\ 6x=48;$$
$$\therefore\ x=8,$$

and $$7x=56.$$

Thus the two numbers required are 8, 56.

Example 2. Divide 24 into two such parts that three times one part added to seven times the other may be 100.

Let x be one part, then $24-x$ is the other; also three times one part is $3x$ and seven times the other is $7\,(24-x)$, and the sum of these numbers is 100.

Hence $$3x+7\,(24-x)=100;$$
$$\therefore\ 4x=68;$$
$$\therefore\ x=17,$$

and $$24-x=7.$$

Thus the two parts of 24 required are 17, 7.

Example 3. Divide Rs. 100 among three persons A, B, C so that A may have Rs. 20 less than B and Rs. 10 more than C.

Let x rupees be A's share, then B's share is $x+20$ rupees and C's is $x-10$ rupees, and the sum of all their shares is Rs. 100.

Hence $$x+x+20+x-10=100;$$
$$\therefore\ 3x=90;$$
$$\therefore\ x=30;$$
$$\therefore\ x+20=50,\quad x-10=20.$$

Thus A's share is Rs. 30, B's is Rs. 50 and C's is Rs. 20.

Example 4. Gopal can perform a piece of work in 42 days and Govind can perform the same piece in 56 days; in how many days can they perform it if they work together?

Let x be the number of days required; then Gopal can perform $\frac{1}{42}$ of the work in one day and therefore $\frac{x}{42}$ of it in x days, and similarly Govind can perform $\frac{1}{56}$ of the work in one day and therefore $\frac{x}{56}$ of it in x days. But in x days they together complete the work; hence

$$\frac{x}{42}+\frac{x}{56}=1;$$
$$\therefore\ 4x+3x=168;$$
$$\therefore\ 7x=168;$$
$$\therefore\ x=24.$$

Thus Gopal and Govind working together can complete the work in 24 days.

Example 5. A father is nine times as old as his son now, but in twelve years will be only three times as old; find the ages of father and son.

Let x years be the son's age, then $9x$ years is the father's age; also in twelve years the son's age will be $x+12$ years and the father's will be $9x+12$ years, and the latter will be three times the former. Hence

$$9x+12=3(x+12);$$
$$\therefore\ 6x=24;$$
$$\therefore\ x=4,$$

and
$$9x=36.$$

Thus the son's age is now 4 years and father's 36 years.

Example 6. A pole 9 feet 11 inches long is to be sawn into two pieces, one of which is six times as long as the other; find the length of each piece.

Let x be the length in inches of the shorter piece, then $6x$ inches is the length of the longer, and the sum of x inches and $6x$ inches must be 9 ft. 11 in., i.e. 119 inches.

Hence
$$x+6x=119;$$
$$\therefore\ 7x=119;$$
$$\therefore\ x=17,$$

and
$$6x=102.$$

Thus the lengths required are 17 inches, 102 inches.

Example 7. Find a number such that if 3, 5, 8 be severally added to it, the product of the least and greatest sums shall be equal to the square of the remaining sum.

Let x be the number, then the three sums formed by adding 3, 5, 8 to it are $x+3$, $x+5$, $x+8$, and by the question the product of $x+3$ and $x+8$ is equal to the square of $x+5$.

Hence $$(x+3)(x+8)=(x+5)^2;$$

$$\therefore\ x^2+11x+24=x^2+10x+25;$$

$$\therefore\ x=1.$$

Thus 1 is the number required, and adding 3, 5, 8 to 1 we get 4, 6, 9; and 36, the product of 4 and 9, is the square of 6.

Example 8. A man remits the same sum in sterling from India every month for the support of his family in England, and he finds that with exchange at 1*s.* 6*d.* he had eighty rupees less to pay than with exchange at 1*s.* 2*d.*; find the amount of his monthly remittance.

Let x pounds sterling be the amount of the man's monthly remittance, then £x are $240x$ pence; and $240x$ pence divided by 18 pence, the cost of one rupee with exchange at 1*s.* 6*d.*, gives the number of rupees which are the equivalent of £x at that rate. Similarly $240x$ pence divided by 14 pence, the cost of one rupee with exchange at 1*s.* 2*d.*, gives the number of rupees which are the equivalent of £x at the lower rate; and the difference of these two quotients is 80 by the question. Hence

$$\frac{240x}{14}-\frac{240x}{18}=80;$$

i.e.
$$\frac{3x}{14}-\frac{x}{6}=1;$$

$$\therefore\ x=21.$$

Thus the man makes a monthly remittance of £21.

Exchange is said to fall when the value of the rupee in sterling is diminished and is said to rise when the value of the rupee in sterling is increased, and exchange is at par when the rupee is worth two shillings. The rate quoted daily in the Money Market and notified in the newspapers is the sterling value of the rupee in shillings and pence and fractions of a penny for that day.

Example 9. At what time between 4 and 5 o'clock are the hands of a watch (1) together, (2) opposite each other?

Each division on the dial of a watch represents one minute in time, and the minute hand describes the whole dial, i.e. sixty of these divisions, while the hour hand describes five, so that the minute hand travels twelve times as fast as the hour hand. Now at 4 o'clock the minute hand is at 12 and the hour hand at 4 and the hands are separated by twenty divisions, and if x be the number of divisions over which the minute hand must then travel in order that the hands may coincide, $\frac{x}{12}$ is the number of divisions over which the hour hand

must travel in the same time. The distance of each hand from 12, measured in divisions, is at the moment of coincidence the same; hence

$$x=\frac{x}{12}+20;$$

$$\therefore\ 11x=240;$$

$$\therefore\ x=21\tfrac{9}{11}.$$

Thus the hands are together at $21\frac{9}{11}$ minutes after 4.

When the hands are opposite each other they are separated by thirty divisions; and, measuring the distance of each hand in divisions from 12 as before, we have

$$x=\frac{x}{12}+20+30;$$

$$\therefore\ 11x=600;$$

$$\therefore\ x=54\tfrac{6}{11}.$$

Thus the hands are opposite each other at $54\frac{6}{11}$ minutes past 4.

The solution of many of these examples might have been obtained by employing more than one unknown quantity; but it is advisable to employ only as many as are necessary, since those which are needlessly introduced have to be eliminated before we can arrive at the solution.

In all problems involving concrete quantities of the same kind care must be taken to express such quantities in terms of the same unit; thus in the 6th example the whole length and the two pieces of the pole should be expressed all in inches or all in feet.

101. *Problems which lead to equations involving two unknown quantities.*

Example 1. A number of two digits is six times the sum of its digits, and if 9 be subtracted from the number its digits are reversed; find the number.

Let x be the digit in the ten's place and y the digit in the unit's place, then $10x+y$ is the number and is six times the sum of x and y. Also when 9 is subtracted from the number we get $10x+y-9$, and the digits are then reversed so that the value of the new number is $10y+x$. Hence

$$10x+y=6(x+y),$$
$$10x+y-9=10y+x;$$

which when simplified become

$$4x-5y=0 \quad \ldots\ldots(1),$$
$$x-y=1 \quad \ldots\ldots(2).$$

Multiplying (2) by 5, $5x-5y=5,$

and subtracting (1), $x=5;$

$\therefore\ y=4.$

Thus the number required is 54.

Example 2. When the numerator and denominator of a certain fraction are each increased by 1 its value becomes $\frac{1}{2}$, and when the numerator and denominator are each diminished by 3 its value becomes $\frac{1}{3}$; find the fraction.

Let the fraction be $\frac{x}{y}$, then when the numerator and denominator are each increased by 1 the fraction becomes $\frac{x+1}{y+1}$, and when the numerator and denominator are each diminished by 3 the fraction becomes $\frac{x-3}{y-3}$.

Hence $$\frac{x+1}{y+1}=\frac{1}{2},$$

$$\frac{x-3}{y-3}=\frac{1}{3};$$

which when simplified become

$$y-2x=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$3x-y=6 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

adding (1) and (2), $x=7;$

$\therefore\ y=2x+1=15.$

Thus the fraction required is $\frac{7}{15}$.

Example 3. The charges for dinner at a restaurant are for a man as much as for a woman and child together, for a woman as much as for a child and one quarter as much as for a man, and for a child twelve annas; find the charge for a man and woman respectively.

Let the charge in annas be x for a man and y for a woman; then x annas, the charge for a man, is equal to the sum of y annas, the charge for a woman, and 12 annas, the charge for a child; and y annas, the charge for a woman, is equal to the sum of $\frac{x}{4}$ annas, one quarter the charge for a man, and 12 annas, the charge for a child. Hence

$$x=y+12,$$

$$y=\frac{x}{4}+12;$$

$$\therefore\ x-12=y=\frac{x}{4}+12;$$

$$\therefore \quad 3x=96;$$

$$\therefore \quad x=32,$$

and $$y=x-12=20.$$

Thus the charge for a man is 32 annas, i.e. 2 rupees, and for a woman is 20 annas, i.e. 1 rupee 4 annas.

Example 4. There are two brothers the sum of whose ages diminished by 4 years is twice the age of the younger, and the difference of whose ages increased by 4 years is one third of the age of the elder; what are their ages?

Let x be the age of the younger and y the age of the elder; then the sum of their ages, $x+y$, diminished by 4, is twice the age of the younger, x; and the difference of their ages, $y-x$, increased by 4, is one third of the age of the elder, y.

Hence $$x+y-4=2x,$$

$$y-x+4=\frac{y}{3};$$

which when simplified become

$$y-x=4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$3x-2y=12 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Multiplying (1) by 2, $$2y-2x=8,$$

and adding this to (2), $$x=20;$$

$$\therefore \quad y=x+4=24.$$

Thus the ages of the brothers are 20 years, 24 years.

Example 5. A shopkeeper requiring change drew from the bank Rs. 540 in silver, the number of 8-anna pieces being twice the number of rupees, the number of 4-anna pieces being equal to the number of rupees and 2-anna pieces, and the number of 2-anna pieces being half the number of 8-anna pieces and 4-anna pieces; find the number of each coin.

Let x be the number of rupees, y the number of 2-anna pieces; then $2x$ is the number of 8-anna pieces, $x+y$ is the number of 4-anna pieces and y, the number of 2-anna pieces, is half the sum of $2x$, the number of 8-anna pieces, and $x+y$, the number of 4-anna pieces. Also the total value of all the coins in annas is 540×16. Hence

$$y=\tfrac{1}{2}(2x+x+y),$$

$$16x+8\times2x+4(x+y)+2y=540\times16;$$

which when simplified become

$$y=3x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$6x+y=1440 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Substituting the value of y from (1) in (2),

$$9x=1440;$$

$$\therefore\quad x=160;$$

$$\therefore\quad 2x=320;$$

also $$x+y=4x=640,$$

and $$y=3x=480.$$

Thus the number of rupees is 160, of 8-anna pieces 320, of 4-anna pieces 640 and of 2-anna pieces 480.

Example 6. A steamer moving at a uniform speed meets with an accident to her machinery which reduces her speed by one-third and causes her to arrive two days late at her destination, the time which would otherwise have been occupied on the voyage being 20 days; if the accident had happened 500 miles nearer to port, she would have arrived only one day late. Find the original speed of the steamer and the whole distance travelled, the distance travelled at the reduced speed being 1000 miles. How many days from port did the accident happen?

Let x be the number of miles per day travelled originally, y the whole distance in miles; then $\frac{2}{3}x$ is the number of miles per day travelled after the accident and $y-1000$ is the distance travelled at the original speed, so that $\frac{y-1000}{x}$ is the number of days' sailing at the original speed and $\frac{1000}{\frac{2}{3}x}$ is the number of days' sailing at the reduced speed, and the total number of days actually occupied over the voyage is 22, since the steamer arrives two days late. Also $y-500$ and 500 are the distances which would have been travelled at the original and reduced speed respectively if the accident had happened 500 miles nearer to port, and in this case the number of days which would have been occupied in sailing at these respective speeds would have been $\frac{y-500}{x}$ and $\frac{500}{\frac{2}{3}x}$, and the total number of days would have been 21. Hence

$$\frac{y-1000}{x}+\frac{1000}{\frac{2}{3}x}=22,$$

$$\frac{y-500}{x}+\frac{500}{\frac{2}{3}x}=21;$$

which on simplification become

$$22x-y=500,$$

$$21x-y=250;$$

subtracting, $$x=250;$$

$$\therefore\quad y=22x-500=5500-500=5000.$$

Also $$\frac{1000}{\frac{2}{3}x}=\frac{3000}{500}=6.$$

Thus the original speed of the steamer was 250 miles per day, the whole distance travelled was 5000 miles and the accident happened 6 days from port.

102. *Problems which lead to equations involving three unknown quantities.*

Example 1. A number of three digits is nineteen times the sum of its digits; if 99 be added to it its digits are reversed, and if 27 be subtracted from it the digits in the ten's and unit's place are interchanged; what is the number?

Let x, y, z be the digits in the hundred's, ten's and unit's place respectively; then the conditions of the problem at once furnish the following equations:

$$100x+10y+z=19\,(x+y+z),$$

$$100x+10y+z+99=100z+10y+x,$$

$$100x+10y+z-27=100x+10z+y;$$

which on simplification become

$$9x-y-2z=0 \quad \ldots\ldots(1),$$

$$z-x=1 \quad \ldots\ldots(2),$$

$$y-z=3 \quad \ldots\ldots(3).$$

From (2) and (3) $\quad y=x+4,\ z=x+1;$

substituting these values of y and z in (1),

$$9x-x-4-2x-2=0;$$

$$\therefore\ 6x=6;$$

$$\therefore\ x=1;$$

then $\quad y=x+4=5,$

and $\quad z=x+1=2.$

Thus the number required is 152.

Example 2. A convict escapes from jail and running uniformly has gone a mile before his escape is discovered; a warder then starts in pursuit at a uniform speed less than that of the convict, and after an hour mounts a horse and galloping uniformly catches the convict in twelve minutes. If the warder had originally increased his speed by one half he would have caught the convict in half an hour, and if he had originally ridden the horse he would have caught him in four minutes. Find the speed of warder, convict and horse, and the distance from the jail at which the capture took place.

Let x, y, z be the speed of the warder, convict and horse respectively in miles per hour; then after an hour the distance of the warder from the jail is x miles and of the convict is $1+y$ miles, and after 12

minutes, i.e. $\frac{1}{5}$ hours more, the distance of the warder from the jail is $x+\frac{z}{5}$ miles and of the convict is $1+y+\frac{y}{5}$ miles, and these distances must be equal since the capture then takes place. Hence

$$x+\frac{z}{5}=1+y+\frac{y}{5};$$

$$\therefore \quad 5x-6y+z=5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again if the warder had originally increased his speed by one half, i.e. had run with the speed $\frac{3x}{2}$, he would have caught the convict in half an hour; hence

$$\frac{1}{2}\times\frac{3x}{2}=1+\frac{y}{2};$$

$$\therefore \quad 3x-2y=4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Also if the warder had originally ridden the horse he would have caught the convict in 4 minutes, i.e. in $\frac{1}{15}$ hours; hence

$$\frac{z}{15}=1+\frac{y}{15};$$

$$\therefore \quad z-y=15 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (2) and (3), $\quad 3x=2y+4,\ z=y+15;$

multiplying (1) by 3, $\quad 15x-18y+3z=15;$

substituting the values of $3x$ and z in this equation,

$$5(2y+4)-18y+3(y+15)=15;$$

$$\therefore \quad 5y=50;$$

$$\therefore \quad y=10.$$

Then $\quad x=\frac{1}{3}(2y+4)=\frac{24}{3}=8,$

and $\quad z=y+15=25.$

Also $\quad x+\frac{z}{5}=1+y+\frac{y}{5}=13.$

Thus the speed of the warder is 8 miles an hour, of the convict 10 miles an hour and of the horse 25 miles an hour, and the capture takes place 13 miles from the jail.

Example 3. The price of brandy is five shillings, of whiskey three shillings and of gin half-a-crown per quart; find how much of each must be taken to form a mixture worth thirteen shillings per gallon, the mixture being such that the quantity of brandy and whiskey is half the quantity of whiskey and gin.

Let 1 quart of the mixture be composed of x quarts of brandy, y quarts of whiskey and z quarts of gin, so that

$$x+y+z=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Then a gallon of the mixture costs 13 shillings so that a quart costs $\frac{13}{4}$ shillings; hence

$$5x+3y+\tfrac{5}{2}z=\tfrac{13}{4},$$

i.e. $$20x+12y+10z=13 \quad \ldots\ldots(2).$$

Also the quantity of brandy and whiskey is half the quantity of whiskey and gin; hence

$$x+y=\tfrac{1}{2}(y+z),$$

i.e. $$2x+y-z=0 \quad \ldots\ldots(3).$$

Adding (1) and (3), $$3x+2y=1 \quad \ldots\ldots(4).$$

Multiplying (1) by 10, $10x+10y+10z=10$;

and subtracting from (2), $$10x+2y=3 \quad \ldots\ldots(5).$$

From (4) and (5) by subtraction,

$$7x=2;$$

$$\therefore\ x=\tfrac{2}{7}.$$

Then $$y=\tfrac{1}{2}(1-3x)=\tfrac{1}{2}(1-\tfrac{6}{7})=\tfrac{1}{14};$$

and $$z=1-x-y=1-\tfrac{2}{7}-\tfrac{1}{14}=\tfrac{9}{14}.$$

Thus $\frac{2}{7}$ of the mixture must be brandy, $\frac{1}{14}$ whiskey and $\frac{9}{14}$ gin.

Example 4. Two men B and C can do a piece of work in $5\frac{1}{7}$ days, C and A can do it in $4\frac{4}{5}$ days, and A and B can do it in $4\frac{4}{17}$ days; in how many days can A, B, C, working separately, do the work?

Let x, y, z be the number of days occupied by A, B, C alone respectively; then in one day A, B, C can do $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ of the whole respectively.

Now B and C together do the whole in $5\frac{1}{7}$ days, i.e. in $\frac{36}{7}$ days, C and A together in $4\frac{4}{5}$ days, i.e. in $\frac{24}{5}$ days, and A and B together in $4\frac{4}{17}$ days, i.e. in $\frac{72}{17}$ days. Hence

$$\frac{36}{7}\left(\frac{1}{y}+\frac{1}{z}\right)=1, \qquad \frac{24}{5}\left(\frac{1}{z}+\frac{1}{x}\right)=1, \qquad \frac{72}{17}\left(\frac{1}{x}+\frac{1}{y}\right)=1;$$

which may be written

$$\frac{1}{y}+\frac{1}{z}=\frac{7}{36}, \qquad \frac{1}{z}+\frac{1}{x}=\frac{5}{24}, \qquad \frac{1}{x}+\frac{1}{y}=\frac{17}{72};$$

adding these three equations and dividing by 2,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{23}{72}.$$

Then $$\frac{1}{x}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\left(\frac{1}{y}+\frac{1}{z}\right)=\frac{23}{72}-\frac{7}{36}=\frac{9}{72}=\frac{1}{8};$$

$$\frac{1}{y}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\left(\frac{1}{z}+\frac{1}{x}\right)=\frac{23}{72}-\frac{5}{24}=\frac{8}{72}=\frac{1}{9};$$

$$\frac{1}{z}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\left(\frac{1}{x}+\frac{1}{y}\right)=\frac{23}{72}-\frac{17}{72}=\frac{6}{72}=\frac{1}{12}.$$

Hence $x=8$, $y=9$, $z=12$; and A, B, C working separately can do the work in 8, 9, 12 days respectively.

EXAMPLES.

Problems to be solved by equations with one unknown quantity.

1. Divide 75 into two parts such that their difference is one-seventh of the less.

2. The sum of two numbers is 24 and the greater is twice the less; what are the numbers?

3. Find two consecutive numbers such that three-fourths of the first is two-thirds of the second.

4. Find three consecutive numbers whose sum shall be 36.

5. The sum of two numbers is eleven times their difference, which is 3; what are the numbers?

6. A certain number is as much below 50 as thrice that number is below 146; find the number.

7. The difference of two numbers is 6 and seven times the greater is ten times the less; find the numbers.

8. If 36 be subtracted from a certain number the result is the same as if twice the number were subtracted from 396; what is the number?

9. Find two consecutive numbers such that the difference of their squares shall be 35.

10. Find a number whose square shall be equal to the product of the two results obtained by adding 3 to it and subtracting 2 from it.

11. Divide 27 into two parts such that a third of one part added to a fifth of the other may make 7.

12. Divide 60 into two such parts that the defect of one part from the whole may be one-third of the defect of the second part from the first.

13. Find three numbers such that the sum of the first and second shall be 7, of the second and third 9 and of the third and first 8.

14. Find a number from the twelfth part of which if 5 be subtracted the remainder is 5.

15. What number is that to which if 148 be added the result is five times the number?

16. Find a number such that the sum of a fifth and a thirteenth of it shall exceed the sum of a seventh and a ninth of it by 94.

17. Divide 56 into four parts such that the excess of the first part above the second, of the second above the third and of the third above the fourth are each equal to 4.

18. Find two consecutive numbers such that the eleventh part of the second shall exceed the twelfth part of the first by 1.

19. A third of one number is a fifth of another number, and the difference of the numbers is 50; find them.

20. Find a number such that the difference of a sixth and a seventh of it exceeds the difference of an eighth and a ninth of it by 30.

21. Divide 88 into two parts so that seventeen times one part shall exceed nine times the other by 14.

22. Divide 31 into two parts so that the square of their difference shall exceed the square of twice the less by 93.

23. There are two numbers differing by 4 and such that the square of the sum of twice the greater and thrice the less is less than the square of five times the greater number by 936; find the numbers.

24. Divide 160 into four parts such that if the first part be increased by 3, the second diminished by 3, the third multiplied by 3 and the fourth divided by 3, the results are all equal.

25. A has three-sevenths of what B has and B has four-ninths of what C has, and in all they have £1133; how much has each?

26. Divide £55 among three persons A, B, C so that A may have six times as much and B four times as much as C.

27. A man paid a bill of five rupees with 4-anna and 2-anna pieces and used in all 33 coins; how many coins of each sort did he use?

28. A man purposing to give a gang of beggars 4 annas each finds that he is 6 annas short; he then gives them 3 annas each and has 9 annas left over; how many beggars were there and how much money had the man?

29. Three brothers divide an inheritance of £600 in such a way that the middle brother receives £50 more than the youngest and £50 less than the eldest; find the shares of the brothers.

30. A man distributed Rs. 43 among 32 old men and women, giving R. 1. 8 a. to each man and R. 1. 4 a. to each woman; how many were there of each sex?

31. A zemindar let two-thirds of his estate and retained in his own hands 81 bighas more than a fourth; find the size of the estate.

32. A is twice as old as B, but eleven years ago he was thrice as old; what is A's age?

33. The ages of a man and his wife are together 52 years, but if the wife's age were doubled it would exceed the husband's by 2 years; find their ages.

34. The difference of the ages of two boys is eight years and is half the sum of their ages; how old are they?

35. A man bequeathed two-thirds of his property to his wife and two-elevenths of the remainder to each of his five children, and there was found to be £120 left; what was the value of his property?

36. A and B have ten rupees between them, but if A were to give B a rupee he would then have double what B had at first; how much has each?

37. Divide £2600 among three persons A, B, C, so that A's share may be the sum of the shares of B and C and six times B's share may be equal to seven times C's share.

38. Divide Rs. 100 into two such parts that one part divided by 7 produces Rs. 7. 8 a. 9 p. and the other part divided by 3 produces Rs. 15. 11 a. 7 p.

39. A subscription of £60 was raised by A, B, C; B subscribed half as much as A and £8 more, and C subscribed half as much as B and £6 more; find the contribution of each.

40. Twenty-one aged persons received a dole; those over 70 years old received 5*s.* each and those between 60 and 70 received 3*s.* 6*d.* each, and a sum of five guineas was distributed; find the number of each class of persons.

41. A man spent a rupee and then borrowed as much as he had left; he again spent a rupee and again borrowed as much as he had left; he finally spent a rupee and found he had nothing left. How much had he at first and how much did he borrow?

42. A costermonger bought a certain number of oranges at two a penny and one-third as many more at three a penny and lost a penny by selling them all at five for twopence; how many oranges were there altogether?

43. A spendthrift ran through a quarter of his fortune in six months, through two-elevenths of the remainder in two months more and through eight-ninths of what was then left in four months more, and after this year's extravagance had only £450 remaining; find the original amount of his fortune.

44. A casual person and a methodical person went on an expedition together, the former starting with twice as much in his purse as the latter; the casual person spent three-sevenths of his money, then mislaid three-quarters of the remainder, then spent half of what remained in his purse, then lost the rest and borrowed five rupees from the methodical person; while the methodical person spent half the amount the casual person did, and after lending him five rupees had ten rupees left. How much had each originally?

45. A wine merchant mixed spirits at three shillings a bottle with spirits at five shillings a bottle and charged £9 for four dozen of the mixture; how many bottles of each kind of spirits were required for the four dozen of the mixture?

46. A man had Rs. 10,000 and invested part at 3 per cent. and the remainder at 4 per cent., and the whole annual interest received was Rs. 370; find the amounts invested at 3 and 4 per cent. respectively.

47. A capitalist invested £600 in two sums which at the rate of $2\frac{1}{2}$ and $3\frac{3}{4}$ per cent. yielded equal interest; how much did he invest at each rate and how much did each sum yield?

48. A lent a certain sum of money to B at 5 per cent. and two-thirds of the same sum to C at $5\frac{1}{2}$ per cent., and the difference of the interest received from B and C was £12; find the total amount lent and the interest received from B and C respectively.

49. A man invested his savings in 4 per cent. Government Rupee Paper at 104 which soon after declined to 99 owing to a fall in exchange; if he had invested after the decline his annual income in rupees from the investment would have been increased by Rs. 20; find the amount invested.

50. Twenty years ago with exchange at par it cost Rs. 1100 less to remit £130 to London from Bombay than it does to-day; what is the rate of exchange to-day?

51. A father of a family residing in India has to make a fixed monthly remittance in sterling to his wife in England, and he finds that the remittance costs him more by Rs. 24 when exchange is at 1*s.* 2*d.* than when the preceding month it was at 1*s.* 3*d.*; what is the amount of his sterling monthly remittance?

52. Exchange drops suddenly and it then costs Rs. 2880 more to remit £3190 to England from India than when exchange stood at the higher rate of 1*s.* $2\frac{1}{2}$*d.*; find the rate of exchange.

53. If a government servant in India with exchange at 1*s.* 2*d.* receive the same pay in rupees as formerly with exchange at par, but if when converted into sterling his pay is £37. 10*s.* per month less now than then, find his pay in rupees.

54. Thirty years ago the rate of exchange was double what it is now, and a remittance of £65 from Bombay to London then cost Rs. 600 less than now; what was the rate of exchange thirty years ago?

55. A traveller from London on landing in Calcutta sold half his English sovereigns at the rate of 1*s.* 4*d.*, and the next week sold the remaining half at the rate of 1*s.* 3*d.*, and he realized altogether Rs. 744; how many sovereigns did he sell?

56. A plank is sawn into two pieces such that seven times one piece is as long as eight times the other, and the shorter piece is found to be 3 feet 6 inches long; what was the length of the plank originally?

57. Find the area of a rectangle whose dimensions are two feet shorter and three feet longer than the side of a square of equal area.

58. Find the dimensions of a rectangle six feet less in breadth than in length, which would have its area diminished by eleven square feet if each side were shortened by a foot.

59. A sugar cane 2 feet 10 inches long is divided among four children in such a way that the first gets 3 inches more than the second, the second 2 inches more than the third and the third 1 inch more than the fourth; find the length of each child's piece.

60. Divide a line 27 yards long into two parts such that the difference of the areas of the squares described upon them shall be equal to the area of a square of side 9 yards.

61. A train travelling at the rate of 20 miles an hour takes 1 hour 12 minutes longer to go from Bombay to Poona than a train travelling at the rate of 25 miles an hour; find the distance between Bombay and Poona.

62. A train leaves A at 11 a.m., passes through B at 11.40 a.m., from this point reduces its speed to $\frac{2}{5}$ of what it had been, and travelling at this diminished speed arrives at C at 1.45 p.m. The distance from B to C being 66 miles, find the distance from A to C.

63. Two cyclists start simultaneously from Bombay and Tanna and travel uniformly, but the one from Bombay at five-sixths of the speed of the one from Tanna, and they meet in half an hour ten miles from Bombay; find the distance of Tanna from Bombay and the speed of each cyclist.

64. A man walked downhill from Mahableshwar to Poona at the rate of 5 miles an hour, but on the return journey uphill he could only walk at three-fifths of his former speed and the journey occupied 9 hours 12 minutes longer; how far is Mahableshwar from Poona?

65. Two boys A and B ran a race; after running uniformly for half a minute and when near the winning-post A was one yard in front of B; B then put on a spurt, and quickening his speed by one-third won by a yard in a second more. Find the speed of each and the length of the course.

66. Two persons started in opposite directions from a point on the equator and travelled uniformly along it, the speed of one being twenty miles a day greater than that of the other, and they met in 80 days; the Earth's circumference being 24,000 miles, find the speed of each.

67. A panther is pursuing a deer and takes two leaps to the deer's three, but one leap of the panther is equal to two leaps of the deer; if the deer have a start of twenty of its own leaps, how many more leaps will it take before it is caught?

68. A dog chases a jackal, running seven yards while the jackal runs six, and catches it in $7\frac{1}{2}$ minutes, the jackal having had a start of 440 yards; find the speed of dog and jackal in miles per hour.

69. When Jamsetjee's mother died her age was treble his, but if ten years had been added to his age her age would only have been double his; how old was Jamsetjee when his mother died?

70. Four times a boy's age is seven times his sister's age, but if two years were added to his age it would then be double his sister's; find the ages of brother and sister.

71. My age, the age of my father and the age of my grandfather are the squares of three consecutive even numbers, and our united ages exceed thrice my age by ninety-two years; how old are we?

72. The Post Office makes three deliveries of letters in the day; by the first delivery I receive thrice as many letters as by the second, and by the third half as many again as by the first and second together, and I receive altogether twenty letters; how many letters do I receive by each delivery?

73. Two men can separately perform a piece of work in seven and eight hours; in what time can they perform it if they work together?

74. *A* can copy 13 pages of a manuscript in 2 hours while *B* can copy 23 pages in 3 hours; how must they divide a manuscript of 85 pages so that they may work for the same time and finish simultaneously?

75. A cistern is supplied with water by two pipes and in 8 minutes can be filled by opening both; if one of the pipes alone could fill the cistern in 12 minutes, in what time could the other fill it?

76. A tank can be filled with water by two pipes singly in 3 hours and $4\frac{1}{2}$ hours, and it can be emptied by a waste pipe in $3\frac{1}{2}$ hours; if all three pipes are open, in what time will the tank be filled?

77. A lady asked her servant what o'clock it was and was answered that it was between 3 and 4 and that the hour and minute hands were together; find what o'clock it was.

78. At what o'clock between 6 and 7 are the hour and minute hands of a watch separated by twenty minute-spaces?

79. A watch has hour, minute and seconds hands moving on the same axis, the seconds hand describing a complete revolution in a minute, and the hands are all together at noon; when will (1) the minute and seconds hands, (2) the hour and seconds hands, be next together?

80. A dishonest milkman adulterated a certain quantity of milk by adding water; if he had put in twice as much water the mixture would have been one-third water; how much pure milk was there in a mixture of five seers?

81. A lady engaged a dirzi for thirty days on condition that for every day he worked he should receive 8 annas and that he should forfeit 2 annas for every day he was absent, and at the end of the time he received 10 rupees; how many days did he work?

82. A ship was provisioned for an eighty days' voyage, but after fifty days one-third of the stores then remaining was condemned and thrown overboard, and at the same time twenty shipwrecked sailors were taken aboard; all on board were then put on half rations and the remaining provisions were made to last out the original time intended; find the number of the ship's crew.

83. Two sailors without means decide to walk overland from Bombay and Calcutta respectively so as to meet; they start simultaneously, the one from Bombay walking at the rate of 5 miles in 2 hours and the other at the rate of 7 miles in 3 hours; they walk 14 hours each day and meet 49 miles nearer Bombay than Calcutta; in how many days do they meet and what is the distance between Bombay and Calcutta?

84. A man starts from Madras on his bicycle for Bombay and travels at a uniform rate nine hours a day; half-way between Madras and Bombay an accident happens to his machine and he has to reduce his speed by one quarter during the remainder of his journey and arrives in Bombay three days later than he had intended; his speed originally being six miles an hour, how many days did his journey occupy and how far is Madras from Bombay by road?

85. The hands of a clock are opposite each other at 6 o'clock; when are they next opposite each other?

86. How many minutes is it past 5 if three quarters of an hour ago it was four times as many minutes past 3?

87. *A*'s watch gains half as much as *B*'s loses; the watches are both adjusted at noon on Monday and on the following Friday at 6 P.M. *A*'s watch is found to be 12 minutes 45 seconds in advance of *B*'s; how many minutes per day do *A*'s and *B*'s watches gain and lose respectively?

88. Two cubical stacks of equal cubical bricks are such that the number of bricks in a horizontal row in a side of one stack is greater by unity than the number of bricks in a horizontal row in a side of the other, and the difference of the numbers of bricks in the stacks exceeds by 31 thrice the square of the number of bricks in a horizontal row in a side of the smaller stack; find the number of bricks in each stack.

89. An officer attempting to form a solid square with his soldiers found he had 39 too many, but that he would require 24 more if each side of the square were to be increased by one man; how many soldiers were there?

90. A colonel draws up his regiment of 1504 men in the shape of a hollow square eight deep; how many men are on each outer side of the square?

91. An officer draws up his men in the form of a hollow square two deep and afterwards three deep, and finds that an inner side in the latter case contains thirty fewer men than in the former; find the number of men.

92. A fruiterer bought Alphonzo mangoes at Rs. 2 per dozen and Mazagon mangoes at R. 1. 12 a. per dozen and sold 39 dozen altogether for Rs. 81, thus realizing $12\frac{1}{2}$ per cent. on his outlay; how many mangoes of each kind did he buy?

93. If the price of 100 lbs. of sugar exceed the price of 8 lbs. of tea by Rs. 3. 13 a. and the price of 3 lbs. of tea exceed the price of 16 lbs. of sugar by R. 1. 9 a. 6 p., find the price of tea.

94. A grocer mixed superior sugar at Rs. 4 per maund with inferior sugar at R. 1. 8 a. per maund and sold the whole at Rs. 2. 8 a. per maund in such a way as to gain 25 per cent. on the cost price; how much superior sugar was in a maund of the mixture?

95. A visitor at Matheran ascended Prabhal Hill at the rate of a quarter of a mile an hour and descended at the rate of seven-twelfths of a mile an hour and occupied $2\frac{5}{7}$ hours over the ascent and descent; what is the height of the mountain in feet?

96. A steamer started from Bombay at 10 A.M. on Dec. 30th, 1893, and went at the rate of 240 miles a day, and she was followed nine hours later by another steamer which went at the rate of 255 miles a day; when and at what distance from Bombay did the second steamer overtake the first?

97. Two men A and B usually drink a cask of beer in 36 days; when both have drank from the cask for nine days A goes away and B then finishes the cask in 63 days; find the time in which A and B would separately drink the contents of the cask.

98. In an examination paper in algebra carrying a hundred marks one student got full marks for bookwork and attempted no rider, while another student, whose marks were two and a half times as many as those of the first, got seven-eighths of full marks on bookwork and thirteen-seventeenths of full marks on riders; how many marks were assigned to bookwork and riders respectively?

99. In an examination 560 marks were assigned to mathematics and 440 to classics and a candidate got 700 marks; if he had answered a quarter more than he did in mathematics and two-fifteenths more than he did in classics he would have got 840 marks; how many marks did he get for mathematics and classics respectively?

100. If the value of gold at a certain rate of exchange is Rs. 28 per tolah and it falls to Rs. 26. 8 a. per tolah on a rise of three farthings in the sterling value of the rupee, what is then the rate of exchange?

Problems to be solved by equations with two unknown quantities.

101. Find two numbers such that their sum shall be 66 and difference 8.

102. The sum of two numbers is 18 and the difference of their squares is 36; find the numbers.

103. Seven times the difference of two numbers is 245 and three times their sum is 459; what are the numbers?

104. Thrice the sum of two numbers is 225 and is five times their difference; find the numbers.

105. Find two numbers such that their sum is seven times their difference and their product is equal to the product of the greater diminished by 6 and less increased by 9.

106. Find a fraction such that if 1 be added to the denominator, or if 1 be subtracted from both numerator and denominator, the value becomes $\frac{1}{2}$.

107. Find a fraction which becomes $\frac{1}{2}$ on adding unity to the denominator and becomes $\frac{1}{13}$ on subtracting unity from the numerator.

108. If the numerator of a certain fraction be increased by $\frac{4}{9}$ and its denominator by $\frac{2}{5}$ its value becomes $\frac{5}{9}$, and if the numerator be diminished by $\frac{7}{8}$ and denominator by $\frac{3}{4}$ its value becomes $\frac{1}{2}$; find the fraction.

109. The sum of the two digits composing a number is 15, and if 9 be added to the number the digits will be reversed; what is the number?

110. A certain number of two digits is seven times the sum of its digits, and by subtracting 27 from it its digits are reversed; find it.

111. The sum and difference of a number of two digits and of the number formed by reversing the digits are 132 and 18 respectively; find the numbers.

112. A number lies between 10 and 100, the sum of its digits is 12, and if the number be divided by the sum of its digits the quotient is 7; what is the number?

113. *A* and *B* had Rs. 200 between them, but *A* spent four-fifths of his money and *B* spent two-thirds of his and then they had only Rs. 50; how much had each originally?

114. A herdsman sold 26 sheep and 28 goats for Rs. 255 and subsequently at the same price 60 sheep and 13 goats for Rs. 382; what was the price of a sheep and a goat respectively?

115. Two boys *A* and *B* compare the contents of their purses and discover that if *A* lent *B* three shillings then *B* would have what *A* had originally, but that if *B* lent *A* a shilling then *A* would have six times what *B* had left; how much had each boy?

116. A man paid a bill of Rs. 26 in rupees, 8-anna and 4-anna pieces; the number of rupees exceeded the number of 8-anna and 4-anna pieces by 6 and was five times the number of 4-anna pieces; how many coins of each kind were used?

117. A bania drew Rs. 550 out of the Bank in notes of ten and five rupees, and the total number of notes exceeded by 12 the eleventh part of the number of rupees drawn out; find the number of notes of each kind which he received.

118. A bill of £2. 11*s.* 6*d.* was paid in half-crowns, florins and shillings; the number of half-crowns was equal to the number of florins and shillings and five times the number of florins exceeded the number of half-crowns and shillings by 10; find the number of each coin.

119. A dole of £17 was distributed among 56 old men and 48 old women and at the same rate might have been distributed among 44 old men and 62 old women; how much did each man and each woman receive?

120. A man has a certain sum in 8-anna, 4-anna and 2-anna pieces, the coins being twenty-two in number; if the number of 8-anna and 4-anna pieces were interchanged he would have Rs. 6. 8 a., and if the number of 4-anna and 2-anna pieces were interchanged he would have Rs. 7. 12 a.; how much money and how many coins of each kind has he?

121. If a lady pays her milkman at Poona Rs. 11. 5 a. 4 p. for 102 seers of milk and in Bombay Rs. 18. 12 a. for 150 seers, and if for 48 quarts she pays Rs. 7 in Poona and Rs. 9 in Bombay, compare the Poona and Bombay seer and find the capacity of each in quarts.

122. A hawker of fruit spent Rs. 5. 12 a. in purchasing oranges and pummeloes, giving two annas for five oranges and five annas for two pummeloes; he then sold three-quarters of his oranges at three annas for five and two-thirds of his pummeloes at seven annas for two and thereby realized as much as he had given for the whole of his stock; how many oranges and how many pummeloes did he buy?

123. The ages of a man and his wife taken together exceed twice the wife's age by sixteen years and the husband's age diminished by a year is thrice the difference of their ages; how old are the husband and wife respectively?

124. The difference of the reciprocals of the figures of a young man's age is half their sum and is $\frac{1}{3}$; how old is he?

125. Double the sum of the ages of a boy and his younger sister is less than their father's age by 5 years, but 6 years ago four times the sum of their ages was less than their father's age by 3 years; the father's age being 49 find the ages of the boy and his sister.

126. *A*'s age is less than the sum of the ages of his two brothers by 19 years, but 22 years ago *A*'s age was double *B*'s and five times *C*'s; how many years ago did *A* come of age and what were then the ages of *B* and *C*?

127. On March 2, 1892, the age of an infant *A* in days was five times as great as that of another infant *B*, but on March 8th was only twice as great; how old were *A* and *B* on March 2nd and on what days were they born?

128. Two men *A* and *B* were engaged to dig an acre of ground; *A* dug by himself for three days and then *A* and *B* dug together for two days and found that a rood was completed; *A* and *B* continued working together for five more days and then *A* went away and *B* finished the remainder in nine days. In how many days could each separately dig the whole ground?

129. A tank which will hold 42000 gallons is filled by two pipes together in 4 hours 40 minutes, but if one pipe were open for 1 hour 12 minutes and then closed and the other were open for 3 hours 45 minutes the tank would be half full; find the quantity of water poured into the tank through each pipe in an hour.

130. *A* walked from Bombay to Bandora ten miles distant at a uniform rate, and starting immediately on the return journey met *B*, who had started from Bombay an hour and a half after him and also walked at a uniform rate, four miles from Bandora; if *B* had started twenty minutes after *A* he would have met *A* two miles from Bandora; find the rates of walking of *A* and *B*.

131. A man started on his tricycle purposing to travel at a uniform speed to a friend's house and back, but after accomplishing a quarter of the return journey he met with an accident and had to drag his machine the remainder of the distance at the speed of two miles an hour, and so arrived at home at the end of $8\frac{1}{2}$ hours instead of 4 hours as intended; find the distance to his friend's house and his speed on the tricycle.

132. A grain merchant sold 2 candies of gram and 16 pharas of bran to a customer for Rs. 60 and at the same rate 7 candies of gram and 40 pharas of bran to another customer for Rs. 205; compare the prices of gram and bran. (8 pharas = 1 candy.)

133. A bania mixed the best rice at 10 annas a pilee with inferior rice at 2 rupees a phara, and sold a candy of the mixture for Rs. 40 at a profit of 25 per cent. on the market price; find the proportion of the two kinds of rice in the mixture. (16 pilees = 1 phara.)

134. If a man buy white sugar at a European shop at Rs. 17. 8 a. per cwt. and brown sugar in the bazar at Rs. 2. 8 a. per maund, in what proportions must he mix the two kinds so as to sell by retail at 1 anna 6 pies per seer and gain 25 per cent. on his outlay?

(1 maund = 28 lbs. = 40 seers.)

135. A grocer bought two sorts of tea at 10 annas and 1 rupee a lb. and mixing them sold the mixture at 14 annas a lb., thus gaining 10 per cent. on his outlay; if he had sold the mixture at 15 annas a lb. he would have gained Rs. 12. 8 a.; find the quantity of each kind of tea.

136. A man invested his savings partly in Railway Guaranteed Stock at 4 per cent. and partly in Mill Shares at $5\frac{1}{2}$ per cent., and the latter investment yielded £51 more than the former the first year but the second year yielded £3 less owing to the interest being reduced to $2\frac{1}{2}$ per cent.; find the amount of stock held in each concern.

137. A person invested a certain sum at $4\frac{1}{2}$ per cent. and a second sum at $3\frac{1}{2}$ per cent., and the interest from the second exceeded that from the first by Rs. 20 per annum; if he had invested the first sum at the rate of interest of the second and the second at the rate of interest of the first, then the interest from the second would have exceeded that from the first by Rs. 140 per annum; find the sum invested at each rate.

138. *A* remits two-thirds of his pay and *B* remits half his pay monthly from India to Europe; when exchange was at 1*s.* 4*d.* *A* and *B* together remitted £86. 13*s.* 4*d.*, and with exchange at 1*s.* 2*d.* *A* remitted £17. 10*s.* more than *B*; find the monthly pay of *A* and *B*.

139. A government officer serving in India has to remit a fixed sterling sum every month to England for the support of his family, and he finds that at first with favourable exchange half his pay sufficed for the purpose, that when exchange fell fivepence lower Rs. 100 more were required, and that when exchange fell threepence lower still five-sixths of his pay was required; what was the amount of his monthly sterling remittance and what was the rate of exchange at first?

140. At a certain rate of exchange a sum of Rs. 24,000 is required for a remittance from Calcutta to London, but if exchange were to rise twopence Rs. 3000 less would suffice; find the sterling sum remitted and rate of exchange.

141. An officer remits to England from Bombay a fixed quarterly sum in sterling for the education of his children, and he finds that one quarter he requires three-eighths as many more rupees as for the preceding quarter when exchange was sixpence higher and Rs. 132 less than the succeeding quarter when exchange was a penny lower; find the amount of his sterling quarterly remittance and the rate of exchange the first quarter. What was the amount in rupees required for the remittance each quarter?

142. An oarsman rows 24 miles down a river in 3 hours and finds that it takes him 12 hours to return; at what rate does the river flow and the man row?

143. A man starts at noon and until 2.30 P.M. canoes down a stream which runs at the rate of 2 miles an hour; he then returns and finds that he is a mile short of his starting-place at 7.15 P.M.; find the distance he went and his rate of canoeing.

144. A man swimming against the stream notices that if he diminish his rate of swimming by half a mile an hour he remains stationary and that if he increase it by a third he covers double his former distance in the same time; find the rate of the stream and of the swimmer.

145. A boatman finds that he can row three miles against the stream in the same time that he can row five miles with it, and that he can row up the stream to the town 15 miles off and back again in eight hours; how long is he in going to the town and how long in returning from it?

146. There are two lines of railway between a town A and another town B, the total length of both lines being 239 miles; by one line a train goes from A to B in $4\frac{1}{4}$ hours, and another train travelling at six-sevenths of the speed of the first train goes from A to B by the other and longer line in 5 hours; find the distance from A to B by each line.

147. A train travelled for 1 hour 30 minutes at a uniform speed, was then detained by a block on the line for 25 minutes, and afterwards accelerating its speed by three-quarters of what it had been before arrived at the terminus 10 minutes late; if the detention had occurred 7 miles nearer the starting point the train, at the accelerated speed, would have arrived only $2\frac{1}{2}$ minutes late. Find the original speed of the train and whole distance travelled.

148. An express train of length 90 feet passes an ordinary train of length 130 feet in 10 seconds when they are moving in the same direction, and in 2 seconds when they are moving in opposite directions; at what speed in miles per hour is each train travelling?

149. A fast train starts from Colaba for Surat half-an-hour after a goods train, travels at the rate of 30 miles an hour, occupies 14 seconds in passing the goods train, stops two hours at Surat, then returns, occupies 6 seconds in passing the goods train and finally arrives at Colaba a quarter of an hour before the goods train reaches Surat; find the speed of the goods train and distance of Surat from Colaba.

150. Lines of railway from A and B cross each other at C, and a train which started at noon from A has a collision at 3 P.M. at the crossing C with a train which started from B at 10 A.M.; if the train from A had increased its speed by a tenth part of the speed of the train from B it would have been three miles beyond the crossing at 3 P.M., and if the train from B had travelled from 2 P.M. to 3 P.M. at the speed of the train from A it would have been seven miles beyond C at 3 P.M.; find the distance of C from A and from B.

151. A greyhound is chasing a hare; the hare takes five leaps to the greyhound's four, but three of the greyhound's leaps are equivalent to four of the hare's; the hare has a start of 40 feet and is caught after taking two hundred leaps; find the length of a leap of the hare and of the greyhound in feet.

152. A company divide the expenses of a dinner and discover that if there had been four more guests each would have paid a rupee less, but that if there had been three fewer each would have paid a rupee more; find the number of guests and contribution of each to the reckoning.

153. A saves one-sixth of his income one month while B spends Rs. 300 more than A and is Rs. 50 in debt at the end of the month; the next month A saves one-third of his income, while B after paying off his debt of the previous month spends Rs. 500 more than A and is Rs. 150 in debt at the end of the month; what is the monthly income of each?

154. Three capitalists, A, B, C compare their fortunes and find that if A were to lend B half a lakh of rupees B would have thrice as much as C, but that if A were to lend C a lakh and a half all three would have equal fortunes; how much has each?

155. If a rectangular lawn were a yard longer and a yard narrower it would contain 82 square yards less, but if it were a yard longer and also a yard wider it would contain 162 square yards more; find its length, breadth and acreage.

156. Three workmen, A, B and C are employed to excavate a trench 720 yards long; in nine days A and B together do 306 yards of which A does 18 yards more than B, and all three can do 49 yards in a day; in how many days could each separately do the whole work?

157. At a Hindu mela a quarter of a lakh of rupees was distributed among mendicants; if there had been 1250 more mendicants each would have received one rupee less, but if there had been 1000 fewer each would have received one rupee four annas more; how many mendicants were there, and how much did each receive?

158. In a two-mile race between two boys, A and B, one boy A has a start of 528 yards but B beats him by 45 seconds, and in a mile race A has a start of 1 minute 30 seconds and there is a dead heat; find the speed of each in miles per hour.

159. In an examination in Mathematics one student got ten-elevenths of full marks for bookwork and seven-eighths of those for problems and obtained 89 marks, while another student got one quarter of full marks for bookwork and one-fourteenth of those for problems and obtained 15 marks; how many marks did the paper carry and how many were assigned to bookwork and problems respectively?

160. In the Euclid and Algebra paper of a University Examination A gets no marks for riders, two-thirds marks for bookwork in Euclid and one-half for bookwork in Algebra, and just passes with the minimum of 25 per cent. marks on the total; while B gets full marks for bookwork, seven-eighths marks for riders in Euclid, and four-fifths for riders in Algebra, and altogether obtains 91 marks out of a total of 100; 53 marks being assigned to Euclid and 47 to Algebra, how many were given to bookwork in Euclid and Algebra respectively?

Problems to be solved by equations with three unknown quantities.

161. Find three numbers such that the sum of the second and third may be 5, of the third and first 4, and of the first and second 3.

162. There are three numbers such that the first is less than the sum of the second and third by 9, the second is less than the sum of the third and first by 7, and the third is less than the sum of the first and second by 1; find the numbers.

163. There are three numbers such that the sum of the first and second is equal to the excess of the third above the second, the sum of the second and third is six times the first, and the sum of the third and first diminished by 1 is thrice the second; what are the numbers?

164. Find three numbers such that the sum of the first and second may be equal to the third, the sum of the first and third may be twice the second, and the excess of the third above the first may be 2.

165. Find three numbers such that the first exceeds the second by as much as the second exceeds the third, and the difference of the squares of the first and second is equal to the sum of the first and second and is 9.

166. The sum of three numbers is 30, the third is five times the difference of the first and second, and the difference of the squares of the first and second is equal to the square of the third; what are the numbers?

167. There are three numbers such that the square of the first is equal to the sum of the squares of the second and third, the third is thrice the difference of the first and second, and the second exceeds the third by 13; find the numbers.

168. There is a certain number of three digits such that if 99 be added to it its digits are reversed, 56 times the number is 45 times the number with the digits reversed, and the sum of the digits is 9; what is the number?

169. Find two numbers each consisting of two digits and having the same right-hand digit, such that if a cypher be inserted between the two digits of each the value of one is increased sevenfold and of the other ninefold, and the sum of the digits of both numbers is 15.

170. There are two numbers of three digits each such that the digits of each are those of the other in reverse order; if the middle digit be omitted from each one number becomes the sixth part of what it was before, while twice the other is nine times what the first has become; the sum of the digits of each number being 9 find the numbers.

171. There is a certain number consisting of three figures whose third figure is equal to the sum of the other two; by interchanging the first and second figures the number is increased by 90 and by interchanging the second and third is increased by 18; find the number.

172. Find three numbers such that the first with eight times the second, the second with twice the third and the third with four times the first may each be equal to 65.

173. There is a number of four digits having 3 for its first digit on the left and 15 for the sum of its digits; if the first and last digits be interchanged the number is increased by 1998, and if the second and third be interchanged the number is increased by 90; find the number.

174. Find a number of three digits which when divided by the sum of its extreme digits produces the quotient 49 and remainder 3, and when divided by six times the middle digit produces the quotient 41; also the sum of its digits divided by the extreme left-hand digit produces the quotient 2 and remainder 4.

175. There are three fractions in descending order of magnitude whose sum is $\frac{2}{3}$; the difference of the first and second is equal to the third, and the difference of the second and third is half the first; find the fractions.

176. A grocer sold to one customer 4 seers of tea, 5 seers of coffee and 12 seers of sugar for Rs. 11. 10 a., to another 7 seers of tea, 3 seers of coffee and 10 seers of sugar for Rs. 13. 4 a., and to a third 5 seers of tea, 6 seers of coffee and 8 seers of sugar for Rs. 13; find the price per seer of tea, coffee and sugar.

177. A lady purchased a cat, a dog and a monkey for a guinea; if the price of the cat had been increased by two-thirds and that of the dog diminished by three-fourths she would still have paid a guinea, and if the price of the dog had been halved and of the monkey doubled she would again have paid a guinea; find how much she gave for the cat, dog and monkey separately.

178. A contractor sold a lot of camels, ponies and mules for Rs. 9750, realizing Rs. 180 for each camel, Rs. 75 for each pony and Rs. 30 for each mule; if the price of each camel had been halved and of each pony doubled he would have realized Rs. 1950 more, and if the price of each pony had been halved and of each mule doubled he would have realized Rs. 525 more; how many of each kind of animal did he sell?

179. A bania sold $2\frac{1}{2}$ candies of gram and $1\frac{1}{2}$ candies of corn to one customer for Rs. 85, $\frac{1}{2}$ candy of gram and 10 pharas of rice to another customer for Rs. 61, and 6 pharas of corn and 7 pharas of rice to a third customer for Rs. 50; find the price per candy of each kind of grain. (8 pharas = 1 candy.)

180. There are three pair of horses worth £800; if a horse of the first pair were interchanged with a horse of the second each pair so formed would be worth £290, but if a horse of the first pair were interchanged with a horse of the third each pair so formed would be worth £260; find the value of each pair.

181. There are three ingots, A, B, C, composed of gold, silver and copper mixed in different proportions; 1 lb. of A contains 1 oz. of gold, 5 oz. of silver and 6 oz. of copper; 1 lb. of B contains 2 oz. of gold, 3 oz. of silver and 7 oz. of copper; and 1 lb. of C contains 4 oz. of gold, 6 oz. of silver and 2 oz. of copper. How much must be taken from each ingot to form a pound of metal containing $2\frac{5}{6}$ oz. of gold, $4\frac{5}{6}$ oz. of silver, and $4\frac{1}{3}$ oz. of copper?

182. Three men, A, B, C, have to dig 180 square yards of ground; B and C can dig 85 yards, C and A can dig 76 yards, and A and B can dig 81 yards in a day; in how many days could each separately complete the whole work?

183. In eight days B and C can earn Rs. 31, C and A Rs. 34, and A and B Rs. 37; how much can each earn in a day?

184. In six hours A, B and C can earn Rs. 7. 2 a., while in four hours A and B can earn Rs. 2. 12 a., and in three hours A and C can earn Rs. 2. 10 a.; what is the pay of each per hour?

185. Three men, A, B, C, undertake a piece of work and perform half of it in 4 days; A then goes away and B and C perform three-fifths of what remains in 4 days; B then goes away and A returns, and A and C complete the remainder in $2\frac{2}{11}$ days; in how many days could each perform the whole work?

186. A tank which will hold 12000 gallons is fed by three pipes; the first and second together can fill it in 1 hour 20 minutes, the second and third together in 2 hours 24 minutes, and the third and first together in 1 hour 30 minutes; how many gallons can each pipe supply singly in an hour?

187. Three pipes together can fill a cistern with water in 20 minutes and a waste-pipe can empty it in 2 hours; the first and second pipes can fill the cistern in 24 minutes and the first and third can fill it in half-an-hour; if the waste-pipe be open in what time can each pipe fill the cistern?

188. A grocer mixed Assam tea at 14 annas per lb. with Ceylon tea at R. 1. 4 a. per lb. and China tea at R. 1. 8 a. per lb. in such a way that the quantity of Assam tea was five-sevenths of the quantity of Ceylon tea and China tea taken together; how much of each sort did he take so as to form a mixture worth R. 1. 2 a. per lb.?

189. A vintner mixes wines which cost him £3, £2. 8*s.* and £1. 16*s.* per dozen in such a way that the quantity of the cheaper wine is thrice the quantity of the other two wines taken together, and he sells the mixture at £2. 4*s.* per dozen at a gain of 10 per cent. on his outlay; how many bottles of each kind of wine must he take to make up a dozen of the mixture?

190. A haystack of uniform rectangular section is such that if its length and breadth were each increased by a foot it would contain 3030 cubic feet more; if its length were diminished by a foot and breadth increased by a foot it would contain 570 cubic feet more; and if its length were increased by a foot and breadth diminished by a foot it would contain 630 cubic feet less; find its dimensions. What is its value if 48 cubic feet are worth a rupee?

191. A man left by will property of the value of £99,000 and he divided it among his six sons, five daughters and widow as follows: each son to have the same share and each daughter to have the same

share, but the share of three sons to be equal to the share of four daughters; and the widow to have half as much again as the share of a son and a daughter together; find the share of the widow, of each son and of each daughter.

192. A man received a sum of Rs. 28 in rupees, eight-anna pieces, four-anna pieces and two-anna pieces; the number of rupees was thrice the number of eight-anna pieces, the number of rupees and four-anna pieces was five-sixths of the number of four-anna pieces and two-anna pieces, and the number of four-anna pieces exceeded the number of eight-anna pieces and two-anna pieces by 1; find the number of each coin.

193. A person paid a bill of Rs. 1000 in notes of twenty rupees, ten rupees and five rupees; the number of ten-rupee notes and five-rupee notes exceeded the number of twenty-rupee notes by 18, while the value of the twenty-rupee notes exceeded the value of the ten-rupee notes and five-rupee notes by a number of rupees which was ten times the number of five-rupee notes; find the number of each kind of note.

194. A grazier bought 24 cows, 16 calves and 600 sheep for £1133; the price of eleven calves was equal to that of ten sheep, and the price of a cow exceeded that of a calf and two sheep by as many threepences as was the total number of animals bought; find the price of a cow, of a calf and of a sheep.

195. My father's age, my age and my son's age are such that my father's age exceeds my age and my son's taken together by five years; twice my age exceeds my father's age and my son's taken together by eleven years; and seven times my son's age exceeds my father's age and mine taken together by five years; what is the age of each of us?

196. Three government servants, A, B, C, regularly remit from India to England for the support of their families seven-eighths, eight-ninths and nine-tenths of their monthly pay respectively; when exchange was at 1*s.* 6*d.* their joint remittance was £180; when exchange declined to 1*s.* 4*d.* A and B together remitted £40 more than C, and when exchange further declined to 1*s.* 3*d.* A and C together remitted £50 more than B; find the monthly pay of each in rupees.

197. Two officers A and B make regular but different sterling remittances every month from Bombay to London; one month their joint remittance cost Rs. 1050, and the following month with exchange a penny lower A's remittance cost Rs. 160 more than B's; also the sum of the two rates of exchange in pence exceeded by 1 the number of pounds sterling remitted monthly by B; find the sterling amount remitted by each and the rates of exchange.

198. A policeman started in pursuit of a dacoit who had ten minutes' start of him and was running uniformly; after running uniformly for half-an-hour the policeman mounted a bicycle and caught the dacoit after cycling two miles; if the policeman had originally increased his speed by a third he would have caught the dacoit in an hour and a

half, and if he had cycled from the first he would have caught him in ten minutes; find the speed of the dacoit and of the policeman when running and when cycling.

199. A hound and a retriever chased a fox which had a start of 440 yards, and the hound caught the fox in three minutes, the retriever then being 704 yards distant; if the hound had diminished his speed by half and the retriever had increased his speed by a third, the retriever would have caught the fox in a quarter of an hour; find in miles per hour the speed of the hound, of the retriever and of the fox.

200. In an examination paper in Algebra valued at 150 marks two students A and B together got twice as many marks in bookwork as a third student C, while C got four times as many marks in problems as A and B together; A and C together got 17 more marks in problems than in bookwork, and B got altogether 6 marks more than A; A, B and C together got 138 marks in bookwork, and C got 4 short of full marks both in bookwork and problems; find how many marks A, B, C individually scored in bookwork and problems respectively, and how many marks were assigned to bookwork and problems separately.

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CHAPTER XIII.

QUADRATIC EQUATIONS.

103. A **Quadratic Equation** is an equation which contains the square of the unknown quantity and no higher power; thus $x^2-3x+2=0$, $4x^2=9$, $ax^2+bx+c=0$ are quadratic equations.

If the square of the unknown quantity alone occur and there is no term of the first degree in the equation, the equation is said to be a **pure** quadratic equation; a pure quadratic equation can at once be solved by extracting the square root, and the two roots can be written down by inspection. Thus if $x^2=36$, extracting the square root we have $x=\pm 6$; if $7(x^2-1)=9(2-x^2)$, the equation may be written $16x^2=25$, and extracting the square root, $4x=\pm 5$, so that $x=\pm\frac{5}{4}$.

If both the first power and the square of the unknown quantity occur in the equation, it is called an **adfected** quadratic equation; thus $x^2+4x-45=0$, $12x^2+x=6$ are adfected quadratic equations.

104. Every quadratic expression when equated to zero forms a quadratic equation. Also the roots of such an equation are the values of the unknown quantity which satisfy the equation, i.e. which make the quadratic expression vanish. Hence when a quadratic expression is capable of resolution into factors by inspection or by methods such as those of Chap. IV. VI., the roots of the equation formed by equating the expression to zero may at once be written down.

Thus $x^2-16=0$ may be written $(x-4)(x+4)=0$, and its roots are $4, -4$; $x^2-7x+10=0$ may be written $(x-2)(x-5)=0$, and its roots are $2, 5$; $8x^2+2x-21=0$ may be written $(2x-3)(4x+7)=0$, and its roots are $\frac{3}{2}, -\frac{7}{4}$, and so on.

Generally if α, β be the roots of the quadratic equation $x^2-px+q=0$, then

$$x^2-px+q=\left(x-\frac{p}{2}\right)^2-\frac{p^2-4q}{4}$$

$$=\left(x-\frac{p}{2}+\frac{\sqrt{p^2-4q}}{2}\right)\left(x-\frac{p}{2}-\frac{\sqrt{p^2-4q}}{2}\right)$$

$$=\left(x-\frac{p-\sqrt{p^2-4q}}{2}\right)\left(x-\frac{p+\sqrt{p^2-4q}}{2}\right)$$

$$=(x-\alpha)(x-\beta);$$

and the expressions x^2-px+q and $(x-\alpha)(x-\beta)$ are thus proved to be identical.

105. The most general form of a quadratic equation is

$$ax^2+bx+c=0.$$

Multiplying by $4a$, transposing the term independent of x and adding b^2 to both sides, we have

$$4a^2x^2+4abx+b^2=b^2-4ac;$$

extracting the square root,

$$2ax+b=\pm\sqrt{b^2-4ac};$$

$$\therefore\quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

This method of solving a quadratic is due to the ancient Hindu mathematicians.

Example. Solve $2x^2-11x+14=0.$

Multiplying by 8, transposing the term independent of x, and adding 121 to both sides, we have

$$16x^2-88x+121=121-112=9;$$

extracting the square root,

$$4x-11=\pm3;$$

$$\therefore\ 4x=11\pm3=14 \text{ or } 8;$$

$$\therefore\ x=\tfrac{7}{2} \text{ or } 2.$$

106. The method most usually adopted is however to *divide by the coefficient of x^2, transpose the term independent of x, and then add to both sides of the equation the square of half the coefficient of x.*

Thus if $$ax^2+bx+c=0;$$

we have by the process indicated,

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2};$$

extracting the square root,

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a};$$

$$\therefore\quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

as before.

Any quadratic equation may be solved either by at once employing formula (1) and assigning to a, b, c the particular numerical values furnished by any particular problem, or by adopting the method of solution here illustrated.

Example. Solve $6x^2-7x+2=0.$

The equation may be written

$$x^2-\tfrac{7}{6}x=-\tfrac{1}{3};$$

adding to both sides the square of half the coefficient of x, i.e. of $-\frac{7}{12}$, we have

$$x^2-\tfrac{7}{6}x+\tfrac{49}{144}=\tfrac{49}{144}-\tfrac{1}{3}=\tfrac{1}{144};$$

extracting the square root,

$$x-\tfrac{7}{12}=\pm\tfrac{1}{12};$$

$$\therefore\ x=\frac{7\pm1}{12}=\tfrac{2}{3}\text{ or }\tfrac{1}{2}.$$

This result may at once be obtained by writing $a=6$, $b=-7$, $c=2$ in formula (1).

107. If the given equation be

$$ax^2+bx+c=0,$$

or

$$x^2+\frac{b}{a}x+\frac{c}{a}=0;$$

we have seen that its roots are given by (1), so that if α, β be these roots,

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a},\qquad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a};$$

hence $$\alpha+\beta=-\frac{b}{a},$$

and $$\alpha\beta=\frac{(-b+\sqrt{b^2-4ac})\,(-b-\sqrt{b^2-4ac})}{4a^2}$$

$$=\frac{b^2-(b^2-4ac)}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}.$$

Thus in any quadratic equation in which the terms are arranged in descending powers of x and the coefficient of the first term is unity, *the sum of the roots is equal to the coefficient of the second term with its sign changed, and the product of the roots is equal to the third term.*

Example 1. Find the sum and difference of the cubes of the roots of the equation $x^2-px+q=0$.

By the proposition just proved, if α, β be the roots of the given equation, we have

$$\alpha+\beta=p, \quad \alpha\beta=q.$$

Hence $$\alpha^3+\beta^3=(\alpha+\beta)\,(\alpha^2-\alpha\beta+\beta^2)$$

$$=(\alpha+\beta)\,\{(\alpha+\beta)^2-3\alpha\beta\}$$

$$=p\,(p^2-3q);$$

and $$\alpha^3-\beta^3=(\alpha-\beta)\,(\alpha^2+\alpha\beta+\beta^2)$$

$$=\sqrt{(\alpha+\beta)^2-4\alpha\beta}\times\{(\alpha+\beta)^2-\alpha\beta\}$$

$$=\sqrt{p^2-4q}\times(p^2-q).$$

Example 2. Construct the equation whose roots shall be the sum and product of the reciprocals of the roots of the equation

$$x^2-px+q=0.$$

With the notation of the previous example the roots of the required equation are to be $\frac{1}{\alpha}+\frac{1}{\beta}$, i.e. $\frac{\alpha+\beta}{\alpha\beta}$, and $\frac{1}{\alpha\beta}$, and these quantities are $\frac{p}{q}$ and $\frac{1}{q}$, so that their sum is $\frac{p+1}{q}$ and product $\frac{p}{q^2}$. Hence the required equation is

$$x^2-\frac{p+1}{q}x+\frac{p}{q^2}=0,$$

i.e. $$q^2x^2-(p+1)\,qx+p=0.$$

108. In the roots

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

of the equation $$ax^2+bx+c=0,$$

if b^2-4ac be positive, the quantity under the radical is positive, and the roots are *possible* and different; if $b^2-4ac=0$, the roots are *equal* and each is $-\frac{b}{2a}$; if b^2-4ac be negative, the quantity under the radical is negative and so is impossible (Art. 76), and the roots are then said to be *impossible*. Thus the roots are possible and unequal if $b^2>4ac$, possible and equal if $b^2=4ac$, and impossible if $b^2<4ac$; also both roots are possible or both are impossible, and one cannot be possible while the other is impossible.

Example 1. Shew that the roots of the equation

$$15x^2+4x-4=0$$

are possible and different.

Here $b^2-4ac=16-4\times15\times-4=256$, a positive quantity; hence the roots are possible and unequal.

Example 2. Find the value of m so that the equation

$$mx^2-12x+9=0$$

may have equal roots.

In the general equation the conditions that the roots shall be equal is $b^2=4ac$, which in this case becomes $144=4\times m\times9=36m$; $\therefore$ $m=4$, and the required equation is

$$4x^2-12x+9=0,$$

and has equal roots, each being $\frac{3}{2}$.

Example 3. Examine the nature of the roots of the equation

$$x^2-2x+2=0.$$

By the method or formula of Art. 106 the roots are found to be $1\pm\sqrt{-1}$ and are impossible, and in this case $b^2-4ac=4-8=-4$ and is negative.

109. Equations sometimes occur which are not strictly quadratics or which are quadratic in form though not in degree, and which, after transformation if necessary, can be solved by the ordinary methods.

Example 1. Solve $4x^4-13x^2+9=0$.

This is an equation of the fourth degree and is at once seen to be equivalent to

$$(x^2-1)(4x^2-9)=0,$$

and thus produces the two quadratic equations $x^2-1=0$, $4x^2-9=0$, the roots of which are 1, -1, and $\frac{3}{2}$, $-\frac{3}{2}$ respectively.

Or we may at once reduce the given equation to an ordinary quadratic $4y^2-13y+9=0$, by writing y for x^2, and may solve for y and subsequently find the corresponding values of x.

Example 2. Solve $x^4-3x^3+4x^2-3x+1=0$.

Dividing by x^2 and re-arranging the terms we have

$$x^2+\frac{1}{x^2}-3\left(x+\frac{1}{x}\right)+4=0;$$

which may be written

$$\left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)+2=0.$$

Put $x+\frac{1}{x}=y$, then the equation becomes

$$y^2-3y+2=0,$$

i.e. $$(y-1)(y-2)=0;$$

the roots of which are 1, 2.

We have now to solve the two equations

$$x+\frac{1}{x}=1, \quad x+\frac{1}{x}=2, \quad \text{i.e.} \quad x^2-x+1=0, \quad x^2-2x+1=0;$$

these are ordinary quadratic equations of which the roots are

$$\frac{1\pm\sqrt{-3}}{2} \text{ and } 1, 1.$$

An equation like that which is here solved, having the coefficients of its terms reckoned from the beginning and end equal, is called a *reciprocal* equation.

Example 3. Solve

$$\frac{x+3}{x+4}+\frac{x-3}{x-4}=\frac{2(x+3)}{x+11}.$$

Clearing of all fractions we have

$$(x+3)(x-4)(x+11)+(x-3)(x+11)(x+4)=2(x+3)(x+4)(x-4),$$

i.e. $x^3+10x^2-23x-132+x^3+12x^2-x-132=2x^3+6x^2-32x-96$;

which becomes on simplification,

$$2x^2+x-21=0,$$

i.e. $$(x-3)(2x+7)=0;$$

so that the roots are 3, $-\frac{7}{2}$.

In this case the equation, which is apparently a cubic, reduces to a quadratic by the terms in x^3 cancelling each other.

As a general rule an equation should be cleared of all fractions before we commence its solution.

Example 4. Solve

$$x^2-4x+\sqrt{x^2-4x+6}=0.$$

The equation may be written

$$x^2-4x+6+\sqrt{x^2-4x+6}-6=0,$$

which, on putting $x^2-4x+6=y^2$, becomes

$$y^2+y-6=0,$$

i.e. $$(y-2)(y+3)=0;$$

and the roots of this equation are 2, -3, so that the corresponding values of y^2 are 4, 9.

We have now to solve the equations

$$x^2-4x+6=4 \text{ or } =9,$$

i.e. $$x^2-4x+2=0, \quad x^2-4x-3=0.$$

These are ordinary quadratic equations of which the roots are $2\pm\sqrt{2}$, $2\pm\sqrt{7}$.

If we now proceed to verify our solution we find that if we restrict the sign of the radical in the original equation to be positive, then the value $y^2=9$ does not satisfy the equation, as in that case we should have $y^2+y=9+3=12$, whereas y^2+y is required to be equal to 6; and in fact the solution $y^2=9$ would satisfy the equation $y^2-y-6=0$ and not $y^2+y-6=0$. Thus the only admissible solution of the given equation would be $x=2\pm\sqrt{2}$.

Example 5. Solve

$$\sqrt{x+6}+\sqrt{3x+7}=\sqrt{16x+1}.$$

Squaring, we have

$$4x+13+2\sqrt{3x^2+25x+42}=16x+1,$$

i.e. $$6x-6=\sqrt{3x^2+25x+42} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Squaring again, $$36x^2-72x+36=3x^2+25x+42,$$

i.e. $$33x^2-97x-6=0.$$

The roots of this equation are 3, $-\frac{2}{33}$, of which the former alone is found by trial to satisfy the given equation, while the root $-\frac{2}{33}$ would satisfy the equation

$$-\sqrt{x+6}+\sqrt{3x+7}=\sqrt{16x+1}.$$

The explanation of this is that when we square (1) we obtain the same result whether the sign of the radical be positive or negative; and so we cannot be certain without trial that the roots finally obtained belong to the given equation or to that equation with the sign of some radical changed.

Simultaneous equations involving more than one unknown quantity.

110. A quadratic equation involving more than one unknown quantity must contain a term or terms of the second degree and no higher; thus $x^2+y^2=25$, $xy=a^2$ are quadratic equations in x and y.

As with simple equations (Art. 94), so with equations involving quadratics, two independent equations are required in order to determine two unknown quantities. These equations may be both quadratic, or one may be simple, or one may be of a higher degree than the second, provided that in all cases they lead to a quadratic equation to be solved. Thus

$$x-y=3, \qquad x^2+y^2=65;$$

or $$x^2-y^2=8, \qquad 2y^2+xy=4;$$

or $$x+y=7, \qquad x^3+y^3=133,$$

which on division lead to the quadratic

$$x^2-xy+y^2=19;$$

are all classed under the general head of simultaneous equations involving quadratics.

Example 1. Solve $x+y=9$ .....(1)

$x^2-xy=56$.....(2).

From (1) $y=9-x$; substituting in (2),

$$x^2-x(9-x)=56,$$

i.e. $$2x^2-9x-56=0,$$

which may be written

$$(x-8)(2x+7)=0;$$

so that $x=8$ or $-\frac{7}{2}$, whence $y=1$ or $\frac{25}{2}$.

Example 2. Solve $xy=15$.....(1),

$x^2+y^2=34$.....(2).

Multiplying (1) by 2, and adding to (2) and subtracting from (2) successively,

$$(x+y)^2=x^2+2xy+y^2=64,$$

$$(x-y)^2=x^2+2xy+y^2=4;$$

extracting the square root, $x+y=\pm 8$,

$$x-y=\pm 2;$$

whence $x=\pm 5$, $y=\pm 3$, or $x=\pm 3$, $y=\pm 5$; x, y having in all cases the same sign.

Example 3. Solve

$$3x^2+xy-y^2=51 \quad \ldots\ldots(1),$$
$$x^2-xy+3y^2=15 \quad \ldots\ldots(2).$$

Write $y=mx$ in both equations, then

$$(3+m-m^2)\,x^2=51,$$
$$(1-m+3m^2)\,x^2=15;$$

by division, $$\frac{3+m-m^2}{1-m+3m^2}=\frac{51}{15}=\frac{17}{5};$$

which, on clearing of fractions, becomes

$$28m^2-11m+1=0,$$

i.e. $$(4m-1)(7m-1)=0;$$

$$\therefore\ m=\tfrac{1}{4} \text{ or } \tfrac{1}{7}.$$

Then (1) by substituting the values of m or $\frac{y}{x}$ becomes

$$x^2=16 \text{ or } \tfrac{49}{3};$$

$$\therefore\ x=\pm 4 \text{ or } \pm\frac{7}{\sqrt{3}}, \text{ and } \therefore\ y=\pm 1 \text{ or } \pm\frac{1}{\sqrt{3}}.$$

Simultaneous homogeneous equations of the second degree of this shape may always be solved by the rule here illustrated; write $y=mx$ in both equations and by division form an ordinary quadratic in m which can be solved by the usual methods, thus leading to the required solution in x and y.

Example 2 might have been solved in this way.

Example 4. Solve

$$x-y=5 \quad \ldots\ldots(1),$$
$$x^3-y^3=335 \quad \ldots\ldots(2).$$

Dividing (2) by (1),

$$x^2+xy+y^2=67;$$

substituting $x-5$ for y from (1),

$$x^2+x(x-5)+(x-5)^2=67,$$

which on simplification becomes

$$x^2-5x-14=0;$$

and the solution of this is $x=-2$ or 7, whence $y=-7$ or 2.

Example 5. Solve

$$\frac{8}{x}-\frac{3}{y}=\frac{1}{2} \quad \ldots\ldots(1),$$
$$\frac{4}{x^2}+\frac{3}{y^2}=1 \quad \ldots\ldots(2).$$

From (1) $\frac{3}{y}=\frac{8}{x}-\frac{1}{2}$; multiplying (2) by 3 and substituting the value of $\frac{3}{y}$, we have

$$\frac{12}{x^2}+\left(\frac{8}{x}-\frac{1}{2}\right)^2=3,$$

which on simplification becomes

$$\frac{304}{x^2}-\frac{32}{x}-11=0.$$

The solution of this is $\frac{1}{x}=\frac{1}{4}$ or $-\frac{11}{76}$, so that

$$\frac{1}{y}=\frac{1}{2} \text{ or } -\frac{21}{38}; \quad \therefore x=4 \text{ or } -\frac{76}{11} \text{ and } y=2 \text{ or } -\frac{38}{21}.$$

In equations of this shape the remarks of Art. 95, Ex. 4 apply; the unknown quantities should be taken to be $\frac{1}{x}, \frac{1}{y}$ and not x, y, as otherwise on clearing the equations of fractions an equation of the fourth degree would result. If preferred the student may at once replace $\frac{1}{x}, \frac{1}{y}$ by x', y' and solve the equations for x', y', returning to the original quantities x, y, in the final results obtained. Similarly if quantities like $\frac{a}{x}, \frac{b}{y}$ are symmetrically involved, it will be convenient to replace them by x', y' and to solve the equations for x', y'.

EXAMPLES.

Solve the quadratic equations:

1. $x^2-5x+4=0$.

2. $x^2-3x-108=0$.

3. $x^2+18x+80=0$.

4. $x^2+2x-195=0$.

5. $3x^2-x-4=0$.

6. $7x^2-9x-36=0$.

7. $12x^2+7x-10=0$.

8. $18x^2-67x+14=0$.

9. $(x-4)^2=36$.

10. $(x+7)^2=169$.

11. $(x-5)^2=x+1$.

12. $4(x-4)^2=9(x-5)^2$.

13. $81(4x-3)^2=(7x+2)^2$.

14. $3(x-2)^2=8x(x-5)$.

15. $9(2x^2-x)=8x+15$.

16. $14x^2-2=3x$.

17. $7x(6-7x)=9$.

18. $6(6x^2-1)=9x+1$.

19. $x-\frac{1}{x}=\frac{15}{4}$.

20. $x+\frac{1}{x}=\frac{10}{3}$.

21. $x-\frac{2}{x}=1.$

22. $x+\frac{1}{x}=2.$

23. $x^2-1=4x.$

24. $x^2=6(x-2).$

25. $(x-2)(x-6)=1.$

26. $(x+11)(x+12)=132.$

27. $(2x+3)(7-5x)=18.$

28. $(6x+1)(3x-2)=7.$

29. $\frac{x+1}{x^2}=\frac{1}{4}.$

30. $\frac{x+4}{x^2-2}=\frac{1}{7}.$

31. $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}.$

32. $\frac{x-1}{x}+\frac{x-3}{x-2}=\frac{2}{3}.$

33. $\frac{1-x}{1+x}+\frac{2-x}{2+x}=\frac{1}{3}.$

34. $\frac{1-2x}{1+2x}+\frac{1-3x}{1+3x}=\frac{5}{6}.$

35. $\frac{1}{x-1}+\frac{1}{x+1}=\frac{9}{4x}.$

36. $\frac{x-1}{x-3}+\frac{2x-1}{x+3}=\frac{28}{x^2-9}.$

37. $\frac{1}{x}-\frac{1}{x+1}=\frac{1}{6(x-1)}.$

38. $\frac{8}{x+3}+\frac{3}{x+2}=\frac{6}{x+1}.$

39. $\frac{3}{x}+\frac{4}{x-2}-\frac{3}{x-4}=0.$

40. $\frac{9}{x+2}-\frac{5}{x+4}-\frac{6}{x+8}=0.$

41. $\frac{8x+8}{9}=\frac{3x^2-3}{4x}.$

42. $\frac{x-60}{x-25}=\frac{2x-60}{3x-10}.$

43. $\frac{3x-2}{4x-3}=\frac{7x-6}{9x-8}.$

44. $\frac{11x+3}{9x+13}=\frac{12x+1}{10x+11}.$

45. $\frac{6x-1}{7x-11}-\frac{4x-5}{5x-9}=\frac{6}{17}.$

46. $\frac{2}{x}+\frac{9}{x+1}=4.$

47. $\frac{x-2}{x-4}=4\frac{x-4}{x-2}.$

48. $\frac{x^2-x+1}{x^2+x+1}=\frac{133}{157}.$

49. $3\frac{x-1}{x-2}+4\frac{x-3}{x-4}=12.$

50. $11\frac{2x-3}{x+3}+10\frac{2x+3}{x-3}=51.$

51. $x^2-33ax+272a^2=0.$

52. $x^2+3ax=154a^2.$

53. $14x^2-ax-4a^2=0.$

54. $39x^2=16a(x+3a).$

55. $(x-a)(x+3a)=(2x-3a)(3x-5a).$

56. $(3x-4a)(4x+3a)=(5x-6a)(6x+a).$

57. $(a+x)^2=a(a-x).$

58. $(a-2x)^2=(3a-4x)^2.$

59. $(x-a)^2+(x+a)^2=(3x-a)^2.$

60. $(4x-3a)^2-(3x-2a)^2=(2x-a)^2.$

61. $x^2-2ax+a^2-b^2=0.$

62. $x^2-(a-b)x-a(2a-b)=0.$

63. $ab(1-x^2)=(a^2-b^2)x.$

64. $(a^2-b^2)x^2=(a^2+b^2)x+ab.$

65. $4x(x-a+b)=(a+b)^2-(a-b)^2$.

66. $5x(5x-2b)=a^2-b^2$.

67. $(b-c)x^2+(c-a)x+a-b=0$.

68. $ax^2-(b+c)x=a+b+c$.

69. $(a^2-b^2-c^2+2bc)x^2=(a+b+c)(2x-a-b-c)$.

70. $a(b+c)(1+x^2)=\{a^2+(b+c)^2\}x$.

71. $\frac{x}{a}+\frac{a}{x}=\frac{1}{a}+a$.

72. $\frac{x}{a}-\frac{a}{x}=\frac{8}{3}$.

73. $\frac{a}{a-x}-\frac{a}{a+x}=\frac{8}{15}$.

74. $\frac{x-a}{x+a}+\frac{2x-a}{2x+a}=\frac{1}{3}$.

75. $\frac{a-x}{a+x}=m^2\frac{a+x}{a-x}$.

76. $\frac{x^2+ax+a^2}{x^2-ax+a^2}=\frac{1}{3}$.

77. $\frac{x-2a}{x-4a}=\frac{3x-2a}{2x-4a}$.

78. $\frac{ax-a^2}{2}=\frac{2}{ax-a^2}$.

79. $\frac{4}{x}-\frac{4}{x+a}=\frac{1}{2x-3a}$.

80. $\frac{m-1}{ma-x}+\frac{n-1}{na-x}=\frac{2}{a}$.

81. $\frac{x^2}{a^2}+\frac{x}{b^2}=2\frac{a^2}{b^4}$.

82. $\frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{a}+\frac{1}{b}$.

83. $\frac{1}{x-a}+\frac{1}{x-b}=\frac{1}{a}+\frac{1}{b}$.

84. $\frac{a}{x+a}+\frac{b}{x+b}=2$.

85. $\frac{b}{x-a}+\frac{a}{x-b}=2$.

86. $\frac{b}{x+a}+\frac{a}{x+b}=\frac{a}{b}+\frac{b}{a}$.

87. $\frac{a}{x-a}+\frac{b}{x-b}=\frac{a}{b}+\frac{b}{a}$.

88. $\frac{a}{x-a}-\frac{b}{x-b}=\frac{(a+b)^2}{(b-a)x}$.

89. $\frac{b^2}{x+a}-\frac{a^2}{x+b}=a-b$.

90. $\frac{x+a}{x-a}+\frac{x+b}{x-b}=2\left(1+\frac{a}{b}+\frac{b}{a}\right)$.

91. $\frac{1}{x-a}+\frac{1}{x-b}=\frac{1}{x-c}$.

92. $\frac{a}{a+x}+\frac{b}{b+x}=\frac{a}{a+c}+\frac{b}{b+c}$.

93. $\frac{a+c}{a+x}+\frac{b+c}{b+x}=2$.

94. $\frac{b^2-c^2}{a-x}-\frac{a^2-c^2}{b-x}=a-b$.

95. $\frac{1}{x-b-c}-\frac{1}{x-c-a}=\frac{1}{a}-\frac{1}{b}$.

96. $\frac{x}{x-a}-\frac{x}{x-b}=\frac{c}{c-a}-\frac{c}{c-b}$.

97. $\frac{c-a}{x-2(c+a)}+\frac{c-b}{x-2(c+b)}=1$.

98. $\frac{b+c-a}{x-4a}+\frac{c+a-b}{x-4b}=1$.

99. $\frac{x-b-c}{x-c}+\frac{x-c-a}{x-c}=\frac{x}{a+b+c}$.

100. $\frac{2a+b+c}{x+a}+\frac{a+2b+c}{x+b}=\frac{2(a+b+c)}{x}$.

Solve the equations:

101. $x^4-13x^2+36=0$.

102. $x^4-5x^2+6=0$.

103. $4x^4-101x^2+25=0$.

104. $324x^4-1017x^2+784=0$.

105. $x^4-17a^2x^2+16a^4=0$.

106. $a^2x^4-2a^3x^2+a^4=0$.

107. $x^6-7x^3-8=0$.

108. $x^6-28x^3+27=0$.

109. $a^2b^2(x^4+1)=(a^4+b^4)x^2$.

110. $a^2b^2c^2x^4=(a^2b^2+c^4)x^2-c^2$.

111. $x^4-x^3-x+1=0$.

112. $x^4-2x^3+2x^2-2x+1=0$.

113. $x^4-6x^3+11x^2-6x+1=0$.

114. $x^4+4ax^3+6a^2x^2+4a^3x+a^4=0$.

115. $4x^8-25x^6+42x^4-25x^2+4=0$.

116. $\dfrac{x-1}{x+1}+\dfrac{x-2}{x+2}=\dfrac{2x+1}{x+7}$.

117. $\dfrac{x}{x-2}+\dfrac{x+1}{x-1}=\dfrac{2x+9}{x}$.

118. $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}+\dfrac{x+3}{x-3}=3$.

119. $\dfrac{x+a}{x+2a}+\dfrac{2x+a}{2x+2a}=\dfrac{4x+11a}{2x+8a}$.

120. $\dfrac{1}{x-6}-\dfrac{2}{x-4}=\dfrac{6}{x+4}-\dfrac{7}{x+6}$.

121. $\dfrac{a-c}{a-x}-\dfrac{b-c}{b-x}=\dfrac{b+c}{b+x}-\dfrac{a+c}{a+x}$.

122. $x^2+x+\dfrac{12}{x^2+x}=8$.

123. $x^2-x+\dfrac{4}{x^2-x}=4$.

124. $(x^2+x+1)^2=4(x^2+x+1)+3$.

125. $(x^2-1)^2=11(x^2-1)+24$.

126. $x^2(2x-5)^2+5x(2x-5)+6=0$.

127. $x^2(1-2x)^2+7x(1-2x)+6=0$.

128. $2(x^2-4x+1)=3-(x^2-4x+1)^2$.

129. $8x(3x+5)=15+(3x^2+5x-1)^2$.

130. $8x(6x-5)=32+(6x^2-5x-2)^2$.

131. $3(x^3-3)=(x^3-3)^2-10$.

132. $x^2-2x+\sqrt{x^2-2x+2}=0$.

133. $x^2-7x-4\sqrt{x^2-7x-4}=0$.

134. $x^2+3x-\sqrt{x^2+3x-6}=8$.

135. $4x^2-4x+3=3\sqrt{4x^2-4x+1}$.

136. $5x^2+x-2=\sqrt{5x^2+x}$.

137. $(x+3)(x+5)=\sqrt{(x+1)(x+7)}+20$.

138. $x+\sqrt{x+7}=5$.

139. $x-3-2\sqrt{x-3}=0$.

140. $2x=15+\sqrt{8x-15}$.

141. $4x=8+3\sqrt{7x+8}$.

142. $\sqrt{7x-6}-\sqrt{5x-1}=1$.

143. $\sqrt{x+1}+\sqrt{x+6}=\sqrt{x+22}$.

144. $\sqrt{4x+5}-\sqrt{3x+1}=\sqrt{7x-6}$.

145. $\sqrt{2x+9}-\sqrt{2x-16}=\sqrt{x-79}$.

146. $\sqrt{(a+b)(a-x)}+\sqrt{(a-b)(a+x)}=2a$.

147. $\sqrt{a^2+b^2+x}+\sqrt{a^2+b^2-x}=2b$.

148. $\sqrt{a+x}-\sqrt{a-x}=\dfrac{2a}{5\sqrt{a-x}}$.

149. $\dfrac{\sqrt{x^2+ax+a^2}}{\sqrt{x^2-ax+a^2}}=\dfrac{\sqrt{x^4+a^2x^2+a^4}}{ax}$.

150. $\dfrac{x-\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}}-\dfrac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}=\dfrac{a^2}{x\sqrt{x^2-a^2}}$.

151. Write down the sum and product of the roots of the equations

$x^2-5x+6=0$, $6x^2-x-2=0$, $20x^2+19x+3=0$, $4x^2-9=0$.

152. State by inspection whether the roots of the following equations are possible and unequal, equal, or impossible:

$18x^2-3x-10=0$, $33x^2+5x-18=0$, $49x^2-84x+36=0$, $5x^2-4x+3=0$.

153. If α, β be the roots of the equation $x^2+px+q=0$, prove that

$$\alpha-\beta=\sqrt{p^2-4q},\quad \alpha^2-\beta^2=-\sqrt{p^4-4p^2q},$$

$$\alpha^2+\beta^2=p^2-2q,\quad \frac{1}{\alpha}+\frac{1}{\beta}=-\frac{p}{q}.$$

154. Find the sum of the squares of the roots of the equations

$x^2-3x-28=0$, $8x^2+14x-15=0$.

155. Find the difference of the squares of the roots of the equations

$x^2-7x+10=0$, $32x^2-20x-63=0$.

156. Find the sum of the cubes of the roots of the equations

$x^2-3x+2=0$, $8x^2+2x-1=0$.

157. Find the difference of the cubes of the roots of the equations

$x^2-7x+10=0$, $2x^2+3x-2=0$.

158. Construct an equation whose roots shall be the sum and product of the roots of the equation $x^2-px+q=0$.

159. Construct an equation whose roots shall be the reciprocals of the roots of the equation $x^2-px+q=0$.

160. Construct an equation whose roots shall be the product of the roots and the sum of the squares of the roots of the equation

$x^2-px+q=0$.

161. Construct an equation whose roots shall be the squares of the roots of the equation $x^2-px+q=0$.

162. Construct an equation whose roots shall be the cubes of the roots of the equation $x^2-px+q=0$.

163. Construct an equation whose roots shall be the sum of the squares and the sum of the cubes of the roots of the equation

$$x^2-px+q=0.$$

164. Construct an equation whose roots shall be the fourth powers of the roots of the equation $x^2-px+q=0$.

165. Shew that the roots of $x^2-272x+7680=0$ are the sum and product of the roots of $x^2-32x+240=0$.

166. Shew that the roots of $3x^2+10x-8=0$ are the reciprocals of the roots of $8x^2-10x-3=0$.

167. Shew that the roots of $x^2-19x-510=0$ are the product and the sum of the squares of the roots of $x^2+2x-15=0$.

168. Shew that the roots of $x^2-130x+3969=0$ are the squares of the roots of $x^2+2x-63=0$.

169. Shew that the roots of $x^2+63x-64=0$ are the cubes of the roots of $x^2+3x-4=0$.

170. Shew that the roots of $x^2-97x+1296=0$ are the fourth powers of the roots of $x^2+x-6=0$.

171. If one of the roots of the equation $x^2-px+q=0$ is n times the other, prove that $np^2=(n+1)^2q$.

172. Shew that one of the roots of $x^2-18x+72=0$ is double the other and that one of the roots of $x^2-16x+48=0$ is treble the other.

173. Find for what values of m the following equations have equal roots: $mx^2-132x+121=0$, $9x^2+2mx+49=0$, $169x^2-390x+m^2=0$.

174. Shew that the difference of the roots of the equation

$$ax^2+bx+c=0$$

is equal to the difference of the roots of the equation

$$a'x^2+b'x+c'=0,$$

if

$$a'^2(b^2-4ac)=a^2(b'^2-4a'c').$$

175. Shew that the roots of

$$x^2-2ax+a^2-b^2-c^2+2bc=0$$

are the sum and difference of the roots of

$$x^2-(a+b+c)x+ab+ca=0.$$

176. Shew that the roots of

$$x^2-2(a^2+b^2)x+a^4-2a^2b^2+b^4=0$$

are the squares of the roots of

$$x^2-2ax+a^2-b^2=0.$$

If α, β be the roots of $ax^2+bx+c=0$:

177. Shew that $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$ are the roots of

$$acx^2-(b^2-2ac)x+ac=0.$$

178. Shew that $\frac{1}{\alpha}+\frac{1}{\beta}$, $\frac{1}{\alpha+\beta}$ are the roots of

$$bcx^2+(b^2+ac)x+ab=0.$$

179. Shew that $\frac{1}{\alpha-3\beta}$, $\frac{1}{\beta-3\alpha}$ are the roots of

$$(16ac-3b^2)x^2-2abx+a^2=0.$$

180. Shew that $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$, $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$ are the roots of

$$c^5x^2+c^2\{b^3-c^3-ac(3b-2c)\}x-b(2ac-b^2)(3ac-b^2)=0.$$

Solve the simultaneous equations:

181. $x+y=12$, $x^2+y^2=74$.

182. $x-y=7$, $xy=44$.

183. $x+y=8$, $x^2-xy=384$.

184. $x-y=3$, $xy+y^2=230$.

185. $x^2+y^2=193$, $xy=84$.

186. $x^2+xy+y^2=3$, $2x^2+3xy+4y^2=9$.

187. $5x^2-3xy+2y^2=20$, $7x^2+4xy-5y^2=7$.

188. $x(x-4y)=48$, $y(3x+y)=76$.

189. $(x-4)(y+3)=143$, $xy=150$.

190. $(x-7)(y-2)=3$, $(x+5)(y+3)=104$.

191. $x(x-5y)=11$, $y(8y-x)=10$.

192. $x(x+y)+y(x-y)=23$, $3x(x+y)=20y(x-y)$.

193. $x+y+1=xy$, $x^2+y^2=13$.

194. $6x-5y=2$, $x^2-3xy+2y^2+5x-4y=12$.

195. $(x+y)^2=48+2(x+y)$, $(x-y)^2=8-2(x-y)$.

196. $x(x+y)=4x+3$, $y(x+y)=4y+2$.

197. $5x-3y=4$, $x+y=xy$.

198. $6x+15y=2xy$, $2x+9y=xy$.

199. $3x+16=7y+5=xy$.

200. $x(x+y)=7x(x-y)=28$.

201. $\frac{x-y}{x+y}=\frac{2}{7}$, $x^2+y^2=106$.

202. $\frac{x^2-y^2}{x^2+y^2}=\frac{3}{5}$, $x-y=3$.

203. $\frac{x-y}{x+y}+\frac{x+y}{x-y}=\frac{17}{4}$, $x^2+y^2=34$.

204. $\frac{x}{x-y}-\frac{y}{x+y}=\frac{5}{3}$, $5(x+y)-8(x-y)=7$.

205. $x-y=7, \quad x^3-y^3=973.$

206. $x+y=4, \quad x^3+y^3=316.$

207. $x^2+xy+y^2=21, \quad x^3-y^3=126.$

208. $x^2-xy+y^2=97, \quad x^3+y^3=1843.$

209. $x^2+xy+y^2=37, \quad x^4+x^2y^2+y^4=481.$

210. $x^2-xy+y^2=109, \quad x^4+x^2y^2+y^4=4251.$

211. $x+y=15, \quad \frac{1}{x}+\frac{1}{y}=\frac{5}{12}.$

212. $x-\frac{1}{y}=1, \quad y-\frac{1}{x}=\frac{1}{12}.$

213. $\frac{1}{x}+\frac{1}{y}=6, \quad \frac{1}{x^2}+\frac{1}{y^2}=20.$

214. $\frac{5}{x}+\frac{11}{y}=4, \quad \frac{7}{x^2}-\frac{13}{y^2}=50.$

215. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x}{a}+\frac{y}{b}=1.$

216. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x}{a}+\frac{y}{b}=2.$

217. $\frac{x}{a}+\frac{y}{b}=a+b, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=a^2+b^2.$

218. $\frac{x}{a}-\frac{y}{b}=1, \quad \frac{x^3}{a^3}-\frac{y^3}{b^3}=7.$

219. $\frac{x^2}{a^2}+\frac{b^2}{y^2}=2, \quad \frac{xy}{ab}=1.$

220. $x(x+y)=a(2a-b), \quad x^2-y^2=b(2a-b).$

221. $2(x-y)=a-b, \quad 4(x^2-xy+y^2)=3a^2+b^2.$

222. $x+y=2a, \quad x^3+y^3=2a(a^2+3b^2).$

223. $a(x-y)=b(x+y), \quad x^2-y^2=ab.$

224. $\frac{8a}{x}-\frac{3b}{y}=1, \quad \frac{16a^2}{x^2}+\frac{90b^2}{y^2}=11.$

225. $\frac{x}{a}-\frac{y}{b}=2c, \quad \frac{xy}{ab}=a^2+b^2-c^2+2ab.$

CHAPTER XIV.

PROBLEMS WHICH LEAD TO QUADRATIC EQUATIONS.

111. The remarks of Art. 98, 99 apply to all equations, irrespective of their degree. The translation of a statement of facts from the language of ordinary life into that of algebra sometimes leads to simple equations, sometimes to quadratics and sometimes to cubics or equations of a higher degree. The present chapter is concerned with problems that produce quadratic equations, which on being solved furnish the values of certain quantities we seek.

It often happens that one of the results obtained is inconsistent with the conditions of the given problem and so is inadmissible; this occurs when the problem is hampered by certain restrictions, inseparable from its nature or from the nature of things, which are not included in the equation at which we arrive. There will be no practical difficulty in discovering whether a result obtained is admissible or not in any particular problem, as the inadmissible result is in such cases manifestly absurd, and generally takes the form of a negative root, while the conditions of the problem render such a value unmeaning.

112. *Problems which lead to equations involving one unknown quantity.*

Example 1. Find what number being added to its square will produce 56.

Let x be the required number, then

$$x^2+x=56,$$

i.e. $$(x-7)(x+8)=0;$$

$$\therefore\ x=7 \text{ or } -8.$$

Thus the number required is 7 or -8; but if, as is often the case, negative numbers are excluded, then the only answer would be 7.

Example 2. Divide 48 into two parts such that their product may be forty-five times their difference.

Let x be the greater part, then $48-x$ is the less, and their difference is $x-(48-x)$, i.e. $2x-48$.

Hence $$x(48-x)=45(2x-48),$$

which on simplification becomes

$$x^2+42x-2160=0,$$

i.e. $$(x-30)(x+72)=0;$$

$$\therefore\ x=30 \text{ or } -72.$$

The former result is alone admissible, as 48 is to be divided into two parts which are *positive* numbers; thus the two parts required are 30, 18.

Example 3. A sum of three rupees was distributed among a gang of beggars; if there had been eight more, each would have received one anna less; how many beggars were there?

Let x be the number of beggars, then each receives $\frac{3\times16}{x}$ annas; and, if there had been eight more, each would have received $\frac{3\times16}{x+8}$ annas; also the former sum exceeds the latter by 1 anna. Hence

$$\frac{3\times16}{x}-\frac{3\times16}{x+8}=1,$$

which on simplification becomes

$$x^2+8x-384=0,$$

i.e. $$(x-16)(x+24)=0;$$

$$\therefore\ x=16 \text{ or } -24.$$

The number of beggars must be *positive* so that the latter result is inadmissible, and the number required is 16.

Example 4. A dealer bought a horse for a certain sum and on re-selling it for £171 gained as much per cent. as was the number of pounds he gave for the horse; find the cost of the horse.

Let x be the number of pounds the dealer gave for the horse; then in return for an outlay of £x he received £171, so that for each pound of outlay he received £ $\frac{171}{x}$ and for each £100 of outlay he received at the rate of £ $\frac{17100}{x}$; his gain per cent. was thus $\frac{17100}{x}-100$. Hence

$$\frac{17100}{x}-100=x,$$

which may be written

$$x^2+100x-17100=0,$$

i.e. $$(x-90)(x+190)=0;$$

$$\therefore\ x=90 \text{ or } -190.$$

The latter root is plainly inadmissible, as the original cost of the horse must have been a *positive* number of pounds; the cost required is therefore £90.

Example 5. If a boy's age and his father's age amount together to 36 years and the eleventh part of the product of their ages exceed the boy's age by 6 years, how old are they?

Let x years be the boy's age, then $36-x$ years is the father's age, and we have

$$\frac{x(36-x)}{11}=x+6,$$

which on simplification becomes

$$x^2-25x+66=0,$$

i.e. $$(x-3)(x-22)=0;$$

$$\therefore\ x=3 \text{ or } 22.$$

The latter root is plainly inadmissible since, if the boy's age were 22, his father's age would then be 14, whereas a son must be younger than his father.

Thus the boy's age must be 3, and therefore the father's is 33.

113. *Problems which lead to equations involving more than one unknown quantity.*

Example 1. Find two numbers whose product shall be 84 and the sum of whose squares shall be 193.

Let x, y be the numbers required; then

$$xy=84 \quad \ldots\ldots(1),$$

$$x^2+y^2=193 \quad \ldots\ldots(2).$$

Multiplying (1) by 2, adding to (2) and subtracting from (2), we have

$$(x+y)^2=361,$$

$$(x-y)^2=25;$$

extracting the square root, $$x+y=\pm 19,$$

$$x-y=\pm 5;$$

$$\therefore\ x=\pm 12,\ y=\pm 7;\ \text{or}\ x=\pm 7,\ y=\pm 12.$$

Example 2. A tank is filled in $\frac{21}{26}$ hours by two pipes which are open together, and the sum of the times occupied by the pipes in singly filling the tank is $3\frac{1}{4}$ hours; in what time would each pipe singly fill the tank?

Let x, y be the number of hours in which the pipes would singly fill the tank, then in 1 hour the pipes would contribute $\frac{1}{x}$, $\frac{1}{y}$ of the whole, and so in $\frac{21}{26}$ hours would contribute $\frac{\frac{21}{26}}{x}$, $\frac{\frac{21}{26}}{y}$ of the whole, and the tank is then full. Hence by the question

$$x+y=1\tfrac{3}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and
$$\frac{\frac{21}{26}}{x}+\frac{\frac{21}{26}}{y}=1,$$

i.e.
$$\frac{1}{x}+\frac{1}{y}=\frac{26}{21} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Dividing (1) by (2), $xy=\frac{21}{8}$;

substituting from (1), $x(1\frac{3}{4}-x)=\frac{21}{8}$,

i.e. $8x^2-26x+21=0$;

which may be written

$$(2x-3)(4x-7)=0;$$

$$\therefore\ x=\tfrac{3}{2} \text{ or } \tfrac{7}{4}, \text{ and } \therefore\ y=\tfrac{7}{4} \text{ or } \tfrac{3}{2}.$$

Thus one pipe fills the tank in $1\frac{1}{2}$ hours and the other in $1\frac{3}{4}$ hours.

PROBLEMS.

Problems to be solved by equations with one unknown quantity.

1. Find a number which shall be less than its square by 156.

2. Find a number which being added to its square produces 306.

3. Find a number which exceeds its square root by 182.

4. The sum of a number and its square root is 650; find the number.

5. The product of a number and of two-thirds of the number exceeds the square of the remaining third by 80; find the number.

6. Five times a number exceeds the square of half the number by 9; find the number.

7. Twice a number is less than the square of a quarter of the number by 20; find the number.

8. The square of half a number exceeds the square of a seventh part of it by 45; find the number.

9. Divide 20 into two parts such that their product shall be five times their sum.

10. Divide 18 into two parts such that their product shall be twelve times their difference.

11. Divide 21 into two parts such that the square of the greater shall exceed ten times the square of the less by 6.

12. Divide 16 into two parts such that the square of one part may be nine times the square of the other.

13. By what number must 3375 be divided so that the quotient may be twice the divisor and the remainder may be 13?

14. Find two numbers whose difference shall be 5 and product 204.

15. Find two numbers whose sum shall be 23 and product 126.

16. Find two numbers whose sum shall be 20 and the sum of the squares 202.

17. Find two numbers differing from each other by 3 and such that their product added to their sum produces 129.

18. Find two numbers which together make up 14 such that their sum shall be less than the sum of their squares by 86.

19. Find two numbers which together make up 12, such that the sum of their squares shall exceed their product by 63.

20. Find two consecutive numbers whose product shall be 306.

21. Find two consecutive odd numbers the sum of whose squares shall be 290.

22. Find two consecutive numbers whose product exceeds their sum by 29.

23. Find two consecutive even numbers such that the difference of their cubes shall be 488.

24. Find three consecutive numbers such that the sum of the squares of the first and second may be equal to the square of the third.

25. Find three numbers such that the second shall be double the first and half the third, and the sum of their squares shall be 189.

26. Find three numbers in ascending order of magnitude and each less than the succeeding number by 3, such that the sum of their squares shall be 93.

27. The sum of the two digits of a number is 15 and their product is less than the number by 22; find the number.

28. A number exceeds five times the product of its two digits by 2 and the left-hand digit exceeds that on the right by the same amount; what is the number?

29. One of the two digits of a number is the square of the other and when 18 is subtracted the digits are reversed; find the number.

30. The difference of a number and its reciprocal is four-fifths of the sum of the number and its reciprocal; find the number.

31. The sum of two fractions is unity and their product exceeds half the smaller fraction by $\frac{1}{16}$; find the fractions.

32. Find two fractions, one exceeding the other by $\frac{7}{40}$, such that their product shall be $\frac{1}{2}$.

33. The sum of two fractions is unity and the difference of their squares is three-fifths of the sum of their squares; find the fractions.

34. The difference of two fractions is $\frac{1}{3}$ and their sum is eight times their product; what are the fractions?

35. The hypotenuse of a right-angled triangle exceeds one side by six inches and the other by a foot; find the dimensions of the triangle.

36. The area of a right-angled triangle is 30 square feet, and the difference of the sides is 7 feet; find the lengths of the sides and of the hypotenuse.

37. The area of a rectangle is twice the area of a square whose side exceeds one side of the rectangle by two inches and is less than the other by a foot; find the dimensions of the rectangle.

38. One side of a rectangular plot of ground exceeds the other by 66 yards and the area is an acre; find the length and breadth of the plot.

39. Divide a line two feet long into two parts such that the rectangle contained by the whole and one part may be equal to the square on the other part, and give your result in inches correct to three places of decimals.

40. A dole of Rs. 72 was distributed among a number of old men; if there had been six fewer, each would have received one rupee more; how many were there?

41. Twenty dozen oranges were distributed equally among a company of children; if there had been twelve more children each child would have received one orange less; how many children were there and how many oranges did each receive?

42. A picnic of ladies and gentlemen was arranged on the understanding that the cost should be equally divided among the men, and it was found that, if a man had taken the place of each of the ten ladies who attended, then each man present would have paid five shillings less; the total cost being £15, find the number of men in the party.

43. Find the price of eggs per dozen when one more for an anna would lower the price by one anna per dozen.

44. What is the price of apples per score if two score less for a shilling would raise the price a halfpenny per score?

45. A contractor sold a number of mules, each for as many rupees as was the number of mules; if he had sold fifteen more for the same money, the sale price of each mule would have been Rs. 20; how many mules were sold?

46. A charitable bania distributed 48 seers of rice among a number of beggars; if each had received 2 seers more, then the number of his seers would have been half the number of beggars; how many beggars were there?

47. A herdsman bought a flock of goats for Rs. 720 and keeping 90 sold the remainder for Rs. 270 at a gain of Rs. 3 each; how many goats did he buy?

48. A lady bought some phulkarries from a borah for Rs. 45; if she had bought one more she would have had twice as many as was the cost in rupees of each phulkarry; how many phulkarries did she buy?

49. Two labourers can dig a piece of ground together in $5\frac{5}{11}$ days, but working singly one would take 2 days longer than the other to perform the work; find the time in which each could singly dig the ground.

50. A tank is supplied with water by two pipes, and by one of them it can be filled 10 minutes sooner than by the other while both together can fill it in $17\frac{1}{7}$ minutes; in what time can each pipe singly fill the tank?

51. Two pedestrians started simultaneously to walk to the same place 21 miles distant; one walked half a mile an hour faster than the other and arrived at his destination an hour before the other; at what rate did each walk?

52. Two horsemen A and B started at the same time to ride 30 miles, and A rode 2 miles an hour slower than B and completed the journey half an hour after B; at what rate did each ride?

53. A wine-merchant sold 4 dozen brandy and 5 dozen whiskey for Rs. 366, and he sold a dozen more of brandy for Rs. 162 than of whiskey for Rs. 60; find the price per dozen of brandy and whiskey.

54. A grocer sells tea 8 annas per lb. dearer than coffee, and 32 lbs. less of tea than of coffee can be purchased for Rs. 77; find the price of tea and coffee.

55. A merchant bought a bale of cloth, and on selling it again for £75 he gained as much per cent. as was the number of pounds the cloth cost; what did the merchant give for the cloth?

56. A grocer bought a quantity of sugar wholesale for Rs. 9 and retailed it at the rate of one rupee per maund more than it cost him, and after repaying himself the original outlay he had 30 seers left over; how many maunds did he buy? (1 maund = 40 seers.)

57. Twenty-six men and boys are paid for their week's work, the men receiving £21 and the boys £21, but each man receives five shillings more than each boy ; how many men were there ?

58. A costermonger bought a number of oranges for 5 shillings and sold all except two dozen, at a farthing each more than they cost, for 6 shillings ; how many oranges did he buy ?

59. The difference of a man's age and his son's age is 28 years, and the eighteenth part of the product of their ages is less than the father's age by 20 years ; find their ages.

60. The sum of the ages of a woman and her daughter is 27 years, and the ninth part of the product of their ages exceeds the daughter's age by 5 years ; how old are they ?

61. A father's age exceeds his son's age by 22 years and his daughter's age by 26 years, and the product of the ages of the father and his two children is twelve times the father's age ; find the ages of the father, son and daughter.

62. If as many hundred pounds are invested as is the rate of interest per cent. and an equal sum is invested at a rate one per cent. lower, and if the two investments together produce £45 per annum, what are the sums invested and at what rates of interest ?

63. A man invested twice as many thousand rupees as was the rate of interest per cent. and twice this sum at a rate one per cent. lower, and the interest from the second investment exceeded that from the first by Rs. 160 ; find the amounts invested and the rates per cent.

64. A government officer remits from India every half year the sum of £195 for the support of his family in England, and he finds that the remittance costs Rs. 480 more when exchange is at a certain rate than when it was twopence higher ; find the rate of exchange.

65. A father of a family makes a remittance of £28 every month from Bombay to his wife in London and finds that formerly with favourable exchange the remittance cost him Rs. 144 less than now with exchange sixpence lower ; what was the rate of exchange formerly and what is it now ?

66. A traveller landing in Bombay sold sixty sovereigns at a certain rate and a month later sold twenty-seven more at a rate three halfpence lower and realized altogether Rs. 1440 ; at what rates did he sell ?

67. To remit to London from Calcutta as many pounds as the number of pence in the rate of exchange increased by unity costs $\frac{64}{75}$ of what it would cost to remit a pound less with exchange threepence lower ; what is the rate of exchange ?

68. The acreage of an estate is twelve times the number of shillings paid as rent for each acre, and, if a forty-eighth part of the estate were laid out as forest and the rent of the remainder were increased by a shilling an acre, the rental would be increased by £3. 10*s.*; what is the extent of the estate and rent per acre?

69. A farmer, by subletting all his land except 40 acres at Rs. 4 per acre more than he paid his landlord, paid his own rent for the whole and had Rs. 80 left over; his rent being Rs. 1600, find the acreage of the farm.

70. The current of a river flows at the rate of 2 miles an hour and it takes a boatman 4 hours to row to a place 6 miles off and to return; at what rate would he row in still water?

71. A man who can swim in a lake at the rate of 140 yards a minute finds that it takes him two minutes longer to swim up a river a distance of 450 yards than to swim the same distance down the river; what is the speed of the current?

72. In consequence of the depreciation of silver the price of whiskey has risen four annas a bottle in India, and a dozen less than formerly can now be purchased for Rs. 330; what is now the price of whiskey per dozen?

73. *A* started from Bombay to walk to Tanna and simultaneously *B* started from Tanna to walk to Bombay and they met in 4 hours, *A* having taken 4 minutes less than *B* to walk a mile; Tanna being 22 miles distant from Bombay, at what speed did each walk?

74. The length of a haystack of uniform rectangular section exceeds the height by 16 ft. and the breadth by 20 ft., and the solid content would remain unchanged if the length were increased by 10 ft. and the breadth by 28 ft. and the height were diminished by 16 ft.; find the dimensions of the haystack.

75. In an examination paper in mathematics carrying a hundred marks a student scored n marks for problems and $n+10$ marks for bookwork and obtained the nth part of full marks on the entire paper; how many marks did he get for problems and bookwork respectively?

Problems to be solved by equations with more than one unknown quantity.

76. Find two numbers such that their product shall be 208 and the sum of their squares 425.

77. Find two numbers such that their sum shall be 20 and the sum of their squares 208.

78. The product of two numbers is 135 and the difference of their squares is 144; find the numbers.

79. The product of two numbers is four times their sum, and the sum of their squares is 425; find the numbers.

80. The product of two numbers is fifteen times their difference, and the sum of their squares is 3744; find the numbers.

81. The sum of two numbers is nine times their difference, and nine times their sum exceeds their product by 4; find the numbers.

82. Find two numbers such that their sum shall be five times their difference and eight times the square of the less shall exceed thrice the square of the greater by 80.

83. Find two numbers such that their product increased by 13 and the sum of their squares diminished by 26 may each be equal to four times their sum.

84. Find two numbers such that four times their product increased by 5 and the sum of their squares diminished by 8 may each be equal to eleven times their difference.

85. The sum of two numbers is 17 and the sum of their cubes is 1853; find the numbers.

86. The difference of two numbers is 7 and the difference of their cubes is 5803; find the numbers.

87. The sum of the cubes of two numbers is 280 and the sum of their squares exceeds their product by 28; find the numbers.

88. The difference of the cubes of two numbers is 316 and the square of their sum exceeds their product by 79; find the numbers.

89. The difference of the fourth powers of two numbers is five times the difference of their squares, and the sum of their squares exceeds their product by 3; find the numbers.

90. The sum of the squares of the two digits of a number exceeds the number by 18, and if 27 be added to the number its digits are reversed; find the number.

91. The product of the two digits of a number is less than the number by 26, and the digits are reversed by subtracting 9 from the number; find the number.

92. Two rectangular fields contain each three acres, but one is 11 yards shorter and 10 yards broader than the other; what are the dimensions of the fields?

93. A reservoir is filled with water in 36 days by two pipes which are open together, but one pipe alone would fill the reservoir 21 days sooner than the other; in how many days would each pipe singly fill the reservoir?

94. Two workmen can together build a wall in 24 hours, but if they work singly it takes one of them 14 hours longer than the other to do the work; find the time in which each man could singly build the wall.

95. Two coolies A and B were engaged at such rates that B's pay for 5 days exceeded A's pay for 4 days by an anna, and at the end of a certain time A, who had worked every day and two days more than B, received Rs. 3. 12 a., while B received Rs. 2. 8 a.; how many days did each work, and what was his rate of pay per day?

96. A boy A can run 880 yards while another boy B runs 704 yards, but if A were to reduce his speed by 110 yards a minute B would then run a mile quicker than A by 20 seconds; what is the speed of each boy?

97. Two persons walk in opposite directions round a courtyard which is of the shape of a right-angled triangle, and, starting from one extremity of the hypotenuse, they meet at the other extremity, one having gone 120 yards more than the other; if one goes 13 yards while the other goes 17 yards, find the dimensions of the courtyard.

98. The fore-wheel of a carriage makes ten more revolutions than the hind-wheel in 150 yards, but if the circumference of each were increased by a quarter of a yard, then the fore-wheel would make nine more revolutions than the hind-wheel in 156 yards; what is the circumference of each wheel?

99. Trains start simultaneously from Bombay and Calcutta, and when they meet one train has travelled 280 miles more than the other and completes its journey in 18 hours 40 minutes, while the other completes its journey 23 hours 20 minutes later; find the rates of the trains and the distance from Bombay to Calcutta.

100. An oarsman who rows 4 miles an hour on still water can row 6 miles up one river 45 minutes quicker than he can row the same distance up another river, while he is $7\frac{1}{2}$ minutes slower in rowing back that distance on the former river than on the latter river; find the rate of the current of each river.

CHAPTER XV.

RATIO. PROPORTION.

114. Ratio is the relation which one quantity bears to another with reference to magnitude and is measured by the number of times one quantity contains the other.

Hence by the definition, the ratio of a to b is represented by the fraction or quotient $\frac{a}{b}$, and the ratio is either expressed in this fractional form or by the notation $a : b$. Thus the ratio of 10 to 2 is $\frac{10}{2}$ or 5 and may also be expressed as 10 : 2 or 5 : 1, while the ratio of 2 to 10 is $\frac{2}{10}$ or $\frac{1}{5}$ and may also be expressed as 2 : 10 or 1 : 5.

The first term of a ratio is called the *antecedent*, and the second the *consequent*, of the ratio; thus in the ratio $a : b$, a is the antecedent and b the consequent. If the antecedent be greater than the consequent, the ratio is said to be of *greater inequality*; if the antecedent be less than the consequent, the ratio is said to be of *less inequality*; and if the antecedent be equal to the consequent, the ratio is said to be a ratio of *equality*. Thus $\frac{4}{3}$ is a ratio of greater inequality, $\frac{3}{4}$ is a ratio of less inequality, and $\frac{4}{4}$ is a ratio of equality.

115. As ratios may be expressed by fractions, therefore they follow the laws of fractions, and so both terms of a ratio may be multiplied or divided by the same quantity without altering the value of the ratio. Thus because the fraction $\frac{ma}{mb}$ is equal to the fraction $\frac{a}{b}$, therefore the ratio $ma : mb$ is the same as the ratio $a : b$.

Also since ratios may be expressed by fractions, therefore they may be compared by reducing their equivalent fractions to a common denominator and by comparing the numerators. Thus the ratios 4 : 7 and 5 : 9 may be expressed by the fractions $\frac{4}{7}$ and $\frac{5}{9}$, i.e. by the fractions $\frac{3\,6}{6\,3}$ and $\frac{3\,5}{6\,3}$ having a common denominator, and the former fraction is the greater, so that the ratio 4 : 7 is greater than the ratio 5 : 9.

The ratio formed by the product of the antecedents of any number of ratios for a new antecedent and the product of the consequents for a new consequent is said to be the *ratio compounded of the given ratios*; thus the ratio *ace* : *bdf* is compounded of the ratios $a : b$, $c : d$ and $e : f$, and the ratio 2 : 5 is compounded of the ratios 3 : 4 and 8 : 15. The ratio $a^2 : b^2$ is called the *duplicate* of the ratio $a : b$, and $a^3 : b^3$ is called the *triplicate* of the ratio $a : b$.

It should be noticed that only *concrete magnitudes of the same kind* can be compared by the method of ratios; thus we can compare rupees with sovereigns but not with inches, and lbs. with oz. but not with days.

116. *By adding the same quantity to each of its terms a ratio of greater inequality is diminished and a ratio of less inequality is increased; and by subtracting the same quantity from each of its terms a ratio of greater inequality is increased and a ratio of less inequality is diminished.*

For let the given ratio be $\frac{a}{b}$; then by adding the same quantity x to both terms of the ratio a new ratio $\frac{a+x}{b+x}$ is formed, and we are to compare the ratios $\frac{a+x}{b+x}$ and $\frac{a}{b}$, i.e. $\frac{b\,(a+x)}{b\,(b+x)}$ and $\frac{a\,(b+x)}{b\,(b+x)}$, which are fractions having a common denominator, so that it suffices to compare their numerators. If $a > b$, then $ax > bx$, $\therefore\ ab + ax > ab + bx$, i.e. $a\,(b+x) > b\,(a+x)$; and if $a < b$, then $ax < bx$, $\therefore\ ab + ax < ab + bx$, i.e. $a\,(b+x) < b\,(a+x)$; hence $\frac{a+x}{b+x} > < \frac{a}{b}$ according as $a > < b$.

Again by subtracting the same quantity x from both terms of the ratio a new ratio $\frac{a-x}{b-x}$ is formed, and we are to compare the

ratios $\frac{a-x}{b-x}$ and $\frac{a}{b}$, i.e. $\frac{b(a-x)}{b(b-x)}$ and $\frac{a(b-x)}{b(b-x)}$. If $a>b$, then $ax>bx$, $\therefore\ ab-bx>ab-ax$, i.e. $b(a-x)>a(b-x)$; and if $a<b$, then $ax<bx$, $\therefore\ ab-bx<ab-ax$, i.e. $b(a-x)<a(b-x)$; hence $\frac{a-x}{b-x}\gtrless\frac{a}{b}$ according as $a\gtrless b$.

Thus $\frac{5}{4}<\frac{4}{3}$ while $\frac{4}{5}>\frac{3}{4}$.

Example 1. What is the least integer which being subtracted from both terms of the ratio 8 : 25 will make it less than the ratio 1 : 8?

In this problem we are to compare the ratios

$$\frac{8-x}{25-x} \text{ and } \frac{1}{8}, \text{ i.e. } \frac{8(8-x)}{8(25-x)} \text{ and } \frac{25-x}{8(25-x)},$$

and we are to find the least integral value of x which will make the former less than the latter. This will happen if

$$64-8x<25-x,$$

i.e. if

$$39<7x,$$

by transposing terms or adding the same quantity $8x-25$ to both sides of the inequality.

Thus we have $7x>39$; $\therefore\ x>5\frac{4}{7}$; and the least integer which is greater than $5\frac{4}{7}$ is 6, which is therefore the integer required.

Example 2. Find two integers in the ratio of 4 to 7 such that if 6 be added to each the ratio becomes $\frac{20}{21}$ of what it would be if 9 were added to each.

Let the two integers be $4x$, $7x$; then by the question,

$$\frac{4x+6}{7x+6}=\frac{20}{21}\times\frac{4x+9}{7x+9};$$

which becomes by simplification,

$$14x^2-51x+27=0,$$

i.e.

$$(x-3)(14x-9)=0.$$

Hence $x=3$, and the required integers are 12, 21; the root $x=\frac{9}{14}$ gives quantities that are not integers.

117. It sometimes happens that the ratio of two quantities cannot be expressed by the ratio of two integers; this occurs for instance when we attempt to compare the diagonal of a square with one side and find the ratio to be $\sqrt{2}$, which is a number that cannot be expressed in the decimal or any other scale though its

value can be assigned to any required deg
the ratio of two quantities cannot be e
ratio of two integers they are said to b
contrariwise when the ratio of two quant
exactly by the ratio of two integers they a
surable. In other words quantities that
terms of a common unit and so have a co
commensurable, whereas quantities that canno
are incommensurable; and a single quantity
incommensurable when it cannot be exactly expr
of a unit.

Thus $\sqrt{3}$, $\sqrt[3]{5}$ are incommensurable quantities.

118. Proportion is the equality of ratios, and four q
tities are said to be proportionals when the ratio of the first to t
second is equal to the ratio of the third to the fourth.

Thus a, b, c, d are proportionals when $\frac{a}{b} = \frac{c}{d}$, and the relation among them is also expressed by the notation $a : b = c : d$, or $a : b :: c : d$. In this case a and d are called the *extremes* and b and c the *means.*

119. *When four quantities are proportionals the product of the extremes is equal to the product of the means.*

For when the four quantities a, b, c, d are proportionals, we have $\frac{a}{b} = \frac{c}{d}$; multiplying by bd we have $ad = bc$, which proves the proposition.

Also conversely *if the product of two quantities be equal to the product of two other quantities, the four quantities are proportionals,* the factors of either product being taken for the extremes and of the other for the means.

For if $ad = bc$, then dividing in succession by bd, cd and ac, we have $\frac{a}{b} = \frac{c}{d}$, $\frac{a}{c} = \frac{b}{d}$, $\frac{b}{a} = \frac{d}{c}$, so that $a : b = c : d$, $a : c = b : d$, $b : a = d : c$.

It follows from this that if four quantities a, b, c, d are proportionals, they are proportionals when taken alternately or

s may also at once be obtained indepen-

n

$$\times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}, \text{ i.e. } \frac{a}{c} = \frac{b}{d};$$

$$1 \div \frac{a}{b} = 1 \div \frac{c}{d}, \text{ i.e. } \frac{b}{a} = \frac{d}{c}.$$

a, b, c, d are proportionals, then

$a + b : b = c + d : d$, and $a - b : b = c - d : d$.

$\frac{a}{b} = \frac{c}{d}$, so that adding unity to both sides we have

$$\frac{a+b}{b} = \frac{c+d}{d}, \text{ i.e. } a + b : b = c + d : d.$$

Also subtracting unity from both sides we have

$$\frac{a-b}{b} = \frac{c-d}{d}, \text{ i.e. } a - b : b = c - d : d.$$

Again if a, b, c, d are proportionals, then (Art. 119) $\frac{b}{a} = \frac{d}{c}$, so that adding unity to both sides and subtracting both sides from unity in succession, we have

$$\frac{a+b}{a} = \frac{c+d}{c} \text{ and } \frac{a-b}{a} = \frac{c-d}{c},$$

i.e. $a + b : a = c + d : c$ and $a - b : a = c - d : c$.

Further since

$$\frac{a+b}{b} = \frac{c+d}{d} \text{ and } \frac{a-b}{b} = \frac{c-d}{d},$$

by division

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}, \text{ i.e. } a + b : a - b = c + d : c - d.$$

121. A particular case of the general proposition of Art. 119 is when b and c are equal, so that $\frac{a}{b} = \frac{b}{d}$ and $ad = b^2$; in this case the product of the extremes is equal to the square of the mean. The quantities a, b and d are then said to be in *continued pro-*

portion, b is called the *mean proportional* between a and d, and d is called a *third proportional* to a and b. Any number of quantities are said to be in continued proportion when every three consecutive quantities are in continued proportion.

Also when a, b and d are in continued proportion so that $\frac{a}{b} = \frac{b}{d}$, multiplying by $\frac{b}{d}$ we have $\frac{a}{d} = \frac{b^2}{d^2} = \frac{a^2}{b^2}$, so that $a : d = a^2 : b^2$. Hence when three quantities are in continued proportion the first bears to the third the duplicate ratio of that which it bears to the second.

122. If $a : b = c : d = e : f$, then each of these ratios is equal to the ratio $a + c + e : b + d + f$.

For $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$ suppose, so that

$$a = kb, \quad c = kd, \quad e = kf;$$

hence by addition $\quad a + c + e = k(b + d + f)$;

$$\therefore \frac{a + c + e}{b + d + f} = k, \quad = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$$

This proves the proposition.

A more general proposition of which the last is a particular case is that each of the given ratios is equal to the ratio

$$pa + qc + re : pb + qd + rf,$$

where p, q, r are any quantities positive or negative.

For we have $\quad pa = kpb, \quad qc = kqd, \quad re = krf$;

hence by addition

$$pa + qc + re = k(pb + qd + rf);$$

$$\therefore \frac{pa + qc + re}{pb + qd + rf} = k, \quad = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$$

The proposition may be extended to any number of ratios.

Example 1. If $\frac{cy - bz}{a^2} = \frac{az - cx}{b^2} = \frac{bx - ay}{c^2}$, prove that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

Dividing numerator and denominator of the first ratio by bc, of the second by ca and of the third by ab, we have

$$\frac{\frac{y}{b} - \frac{z}{c}}{\frac{a^2}{bc}} = \frac{\frac{z}{c} - \frac{x}{a}}{\frac{b^2}{ca}} = \frac{\frac{x}{a} - \frac{y}{b}}{\frac{c^2}{ab}};$$

and by the proposition just proved each of these ratios is equal to a ratio whose numerator is the sum of the numerators and denominator the sum of the denominators; hence each ratio

$$=\frac{\frac{y}{b}-\frac{z}{c}+\frac{z}{c}-\frac{x}{a}+\frac{x}{a}-\frac{y}{b}}{\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}}=0.$$

Therefore $\frac{y}{b}-\frac{z}{c}=0$, $\frac{z}{c}-\frac{x}{a}=0$, $\frac{x}{a}-\frac{y}{b}=0$; i.e. $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

Example 2. Solve the simultaneous equations: $ax+by+cz=0$, $a'x+b'y+c'z=0$, $a''x+b''y+c''z=d$.

From the first two equations we have, by eliminating x, y, z successively,

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b};$$

and by the proposition of this Article each of these ratios

$$=\frac{a''x+b''y+c''z}{a''(bc'-b'c)+b''(ca'-c'a)+c''(ab'-a'b)}$$

$$=\frac{d}{a''(bc'-b'c)+b''(ca'-c'a)+c''(ab'-a'b)}$$

by the third equation; thus the values of x, y, z are determined and the equations are completely solved.

Equations solved in this way are said to be solved *by the method of ratios.*

EXAMPLES.

1. Arrange the following ratios in order of magnitude: 7 : 8, 11 : 12, 8 : 9, 17 : 18.

2. Arrange the following ratios in order of magnitude: 11 : 16, 7 : 10, 13 : 20, 5 : 8, 27 : 40.

3. What ratio is compounded of the ratios 3 : 5 and 10 : 27?

4. Find the ratio compounded of the ratios 3 : 8, 4 : 9 and 6 : 7.

5. Write down the duplicate of the ratio 4 : 9 and the triplicate of the ratio 3 : 5.

6. Find the ratio of six annas to three rupees.

7. Find the ratio of thirty-six rupees to three guineas when exchange is at 1*s.* 2*d.*

8. Find the ratio of 1 foot 6 inches to 10 yards.

9. Find the ratio of 1320 square yards to an acre.

10. What is the ratio of 140 lbs. to 5 cwt.?

11. What is the ratio of 144 minutes to 2 days?

12. What value of x will make the ratio $4+x : 9+x$ become the ratio 3 : 4?

13. What value of x will make the ratio $17+x : 5+x$ become the ratio 7 : 3?

14. What value of x will make the ratio $x-11 : x-7$ equal to the ratio 2 : 3?

15. Find a number which being added to both terms of the ratio 7 : 10 will produce the ratio 6 : 7.

16. Find a number which being subtracted from both terms of the ratio 25 : 21 will produce the ratio 2 : 1.

17. Find x so that the ratio $x+4 : x+17$ may be the duplicate of the ratio 6 : 7.

18. Find x so that the ratio 1 : 4 may be the duplicate of the ratio $x-5 : x-3$.

19. What value of x will make the ratio $1 : x$ the duplicate of the ratio $x : 64$?

20. What value of x will make the ratio $x : 16$ the triplicate of the ratio $81 : x$?

21. What value of x will make the ratio 1 : 8 the triplicate of the ratio $x+1 : x+3$?

22. Find the ratio of y to x so that the ratio $8y+3x : 5y-6x$ may be equal to the ratio 3 : 1.

23. Find the ratio of y to x so that the ratio $11x-4y : 13x+8y$ may be equal to the ratio 1 : 5.

24. Find the ratio of y to x so that the ratio $x^2+2y^2 : 3xy$ may be a ratio of equality.

25. Find the ratio of y to x so that the ratio $xy : 3x^2+10y^2$ may be equal to the ratio 2 : 23.

26. Find the ratio of y to x so that the ratio $6y^2+11xy : x^2$ may be equal to the ratio 2 : 1.

27. Find the least integer which being added to both terms of the ratio 6 : 17 will make it exceed the ratio 3 : 7.

28. Find the least integer which being subtracted from both terms of the ratio 12 : 25 will make it less than the ratio 1 : 6.

29. What is the greatest integer which being added to both terms of the ratio 17 : 8 will make it exceed the ratio 5 : 3?

30. What is the greatest integer which being subtracted from both terms of the ratio 23 : 10 will make it less than the ratio 7 : 2?

31. Find two numbers in the ratio of 5 to 8 such that if 4 be added to each the ratio becomes that of 7 to 10.

32. Find two numbers in the ratio of 3 to 4 such that if 5 be subtracted from each the ratio becomes that of 2 to 3.

33. Find two numbers in the ratio of 5 to 9 such that if each be subtracted from 30 the ratio becomes that of 5 to 1.

34. Find two numbers in the ratio of 9 to 4 such that if 5 be added to the greater number and 4 to the less the ratio becomes that of 2 to 1.

35. Find two numbers in the ratio of 6 to 11 such that if 10 be added to the less number and 15 be subtracted from the greater the ratio becomes one of equality.

36. Find two numbers in the ratio of 2 to 3 such that if 6 be subtracted from each the ratio becomes two-thirds of what it would be if 6 were added to each.

37. Find two integers in the ratio of 8 to 13 such that if 10 be added to each the ratio becomes eight-ninths of what it would be if 35 were added to each.

38. Find two integers in the ratio of 3 to 4 such that if 8 be subtracted from each the ratio becomes fifteen-sixteenths of what it would be if 6 were subtracted from each.

39. Find two consecutive even numbers such that the ratio of their squares shall be the ratio of 4 to 9.

40. Find two consecutive numbers such that the square of the first increased by 7 bears a ratio of equality to the square of the second diminished by 8.

41. Find two numbers in the ratio of 2 to 3 such that their sum bears to the sum of their squares the ratio of 5 to 52.

42. Find two numbers in the ratio of 5 to 3 such that their difference bears to the difference of their squares the ratio of 1 to 96.

43. Find two numbers in the ratio of 4 to 1 such that their sum bears to their product the ratio of 1 to 20.

44. Find two numbers such that their difference, sum and product may be in the ratio 1 : 4 : 15.

45. Find two numbers such that their product, the difference of their squares and the sum of their squares may be in the ratio 3 : 8 : 10.

46. If a and b be unequal quantities prove that the ratio $a^2+b^2 : 2ab$ is always a ratio of greater inequality.

47. Prove that the ratio $a+b : a^2+b^2$ is greater than the ratio $a^2+b^2 : a^3+b^3$ except when $a=b$.

48. What quantity must be added to each term of the ratio $a : b$ so as to produce the ratio $ma : nb$?

49. If the ratio $a : c$ be the duplicate of the ratio $b-a : c-b$, shew that $b^2=ac$.

50. If $a : b$ be a ratio of less inequality and $c : d$ be a ratio of greater inequality, prove that the ratio $a+c : b+d$ is greater than the ratio $a : b$ and less than the ratio $c : d$.

Find what integral values of x will satisfy the following ten proportions:

51. $x : 12=13 : 4$.

52. $7 : 4=x : 8$.

53. $63 : x=x : 7$.

54. $x+1 : x+4=x+6 : x+14$.

55. $x-3 : x-7=x-8 : x-11$.

56. $3x+2 : 4x+1=5x+8 : 7x+4$.

57. $2x-1 : 4x-5=3x-4 : 5x-4$.

58. $3x-2 : 4x+9=5x-2 : 8x+4$.

59. $x-1 : x-3=3(x+3) : 2(x+1)$.

60. $x-1 : x+1=7(x^2-x+1) : 9(x^2+x+1)$.

Find what proportions are deducible from the following five equations:

61. $x^2+y^2=ax$.

62. $x^2-y^2=ay$.

63. $x^3+y^3=a^3$.

64. $x^3-y^3=a^2x$.

65. $x^4+a^2x^2+a^4=y^4$.

If $a : b=c : d$, prove the following five results:

66. $a(b+d)=b(a+c)$.

67. $a^2(b^2-d^2)=b^2(a^2-c^2)$.

68. $(a^2-c^2)(b^2-d^2)=(ab-cd)^2$.

69. $(a+b+c+d)d=(b+d)(c+d)$.

70. $(a+b+c+d)(a-b-c+d)=(a+b-c-d)(a-b+c-d)$.

If a, b, c be in continued proportion, prove the following five results:

71. $c(a-b)=b(b-c)$.

72. $b(a+b+c)=bc+ca+ab$.

73. $(a+b+c)(a-b+c)=a^2+b^2+c^2$.

74. $(a-b)(b+c)=(a+b)(b-c)$.

75. $(a^2-b^2)(b^2-c^2)=b^2(a-c)^2$.

76. If a, b, c, d be in continued proportion, shew that

$$(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2.$$

77. Find three numbers in continued proportion such that their sum shall be 21 and the ratio of the first to the third 1 : 4.

78. Find three numbers in continued proportion such that fifteen times the sum of the extremes shall be seventeen times their difference and the mean shall be 8.

79. Find three numbers in continued proportion such that twice the sum of the extremes shall be five times the mean, and the difference of the extremes shall be 12.

80. Find three numbers in continued proportion such that their sum shall be 13 and the sum of their squares 91.

81. Find four numbers which are proportionals such that the first is less than the second, third and fourth by 9, 16 and 37 respectively.

82. Find four numbers which are proportionals such that the sum of the first and second is 28, of the first and third 40 and of the third and fourth 84.

83. Find four numbers which are proportionals such that the ratio of the first to the second is 1 : 2, the sum of the numbers is 108 and the difference of the first and third is 6.

84. Find four numbers which are proportionals such that the ratio of the first to the third is 1 : 3, the sum of the first and second is 14 and the difference of the third and fourth is 18.

85. Find four numbers which are proportionals such that the product of the extremes is 210, the ratio of the first to the second is 2 : 3 and to the third is 5 : 7.

86. A rod is divided into two parts in the ratio of 5 to 6 and from the same end in the ratio of 7 to 8, and the distance between the points of section is an inch; what is the length of the rod?

87. The sides of a triangular field are in the ratio 8 : 15 : 17, and the area of the field is 8640 square yards; find the lengths of the sides.

88. A man has 550 rupees, eight-anna pieces and four-anna pieces, the sums of money in each denomination being in the ratio 12 : 3 : 1 respectively; what is the number of each kind of coin?

89. A banker pays a constituent the sum of Rs. 1500 in notes of fifty rupees, twenty rupees and ten rupees, the numbers of the notes being in the ratio 5 : 2 : 1 respectively; find the amount paid in each kind of notes.

90. Two bottles contain mixtures of spirits and water; a mixture consisting of one pint from the first bottle and two from the second contains spirits and water in the ratio 29 : 7, while a mixture consisting of one pint from the first bottle and three from the second contains spirits and water in the ratio 13 : 3; what is the proportion of spirits in each bottle?

91. If $\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}$, prove that $x+y+z=0$.

92. If $a+b+c$ is not zero and $\frac{y-z}{a}=\frac{z-x}{b}=\frac{x-y}{c}$, prove that

$$x=y=z.$$

93. If $\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}$, prove that

$$\frac{a}{y+z-x}=\frac{b}{z+x-y}=\frac{c}{x+y-z}.$$

94. If $\frac{x}{b-2c}=\frac{y}{c-2a}=\frac{z}{a-2b}$, prove that

$$\frac{a}{2x+4y+z}=\frac{b}{2y+4z+x}=\frac{c}{2z+4x+y}.$$

95. If $\frac{x}{2(b+c)-a}=\frac{y}{2(c+a)-b}=\frac{z}{2(a+b)-c}$, prove that

$$\frac{a}{2(y+z)-x}=\frac{b}{2(z+x)-y}=\frac{c}{2(x+y)-z}.$$

By the method of ratios solve the equations:

96. $5x-2y+4z=0,\quad 2x-y+z=0,\quad x+y+z=4.$

97. $2x-2y+z=0,\quad 9x+y-3z=0,\quad x^2+y^2+z^2=26.$

98. $ax=by=cz,\ x+y+z=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$

99. $\frac{x}{2b-c}=\frac{y}{2c-a}=\frac{z}{2a-b},\quad 2(x+y+z)=a+b+c.$

100. When four numbers are proportionals, shew that the greatest and least are together greater than the other two.

MISCELLANEOUS EXAMPLES.

CHAP. I.—XII.

UNIVERSITY OF BOMBAY.

1. If $a=2$, $b=3$, $c=4$, find the value of

$$\frac{a-b+c}{a+b-c}+\frac{b-c+a}{b+c-a}+\frac{c-a+b}{c+a-b}.$$

2. Simplify

$$(b+c-a)^3+(c+a-b)^3+(a+b-c)^3+24abc.$$

3. Shew that $\dfrac{x}{x+y}+\dfrac{y}{x-y}=\dfrac{x}{x-y}-\dfrac{y}{x+y}$.

4. Solve the equations:

$$\frac{2x}{3}+\frac{x}{4}=5 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{3x-13}{7}+\frac{11-4x}{3}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

5. A man whose age is 40 has a son who is 9 years old; when will the father be twice as old as his son?

6. A farmer sold to one person 9 horses and 7 cows for Rs. 3000 and to another 6 horses and 13 cows at the same prices and for the same sum; what was the price of a horse and of a cow?

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7. If $a=4$, $b=3$, $c=2$, evaluate

$$\frac{2a+b\,(2c-a)}{3b-\sqrt{2c^3}}.$$

8. Multiply $x^2-\frac{1}{2}x+\frac{2}{3}$ by $\frac{1}{3}x+2$, and a^m-2c^n by $a-c^n$.

9. Divide a^4-81 by $a-3$.

10. Extract the square root of

$$\frac{x^4}{4}+\frac{2x^3}{3}+\frac{4x^2}{9}+2+\frac{8}{3x}+\frac{4}{x^4}.$$

11. Find the H.C.F. of $6a^2+7ax-3x^2$ and $6a^2+11ax+3x^2$, and deduce the H.C.F. of $72a^3x+84a^2x^2-36ax^3$ and $24a^2x+44ax^2+12x^3$.

12. Reduce to their simplest forms:

$$\frac{7x-10}{5}-\frac{3x-7}{6}-\frac{27x-30}{30} \quad \text{.....(1).}$$

$$2\,\frac{x^2-\frac{1}{4}}{2x+1}+\tfrac{1}{2} \quad \text{.....(2).}$$

$$\frac{3}{4(1-x^2)}+\frac{3}{8(1-x)}+\frac{1}{8(1+x)}-\frac{1-x}{4(1+x^2)} \quad \text{.....(3).}$$

$$\frac{x^{3n}}{x^n-1}-\frac{x^{2n}}{x^n+1}-\frac{1}{x^n-1}+\frac{1}{x^n+1} \quad \text{.....(4).}$$

13. Solve the equations:

$$\cdot 15x+\frac{\cdot 135x-\cdot 225}{\cdot 6}=\frac{\cdot 36}{\cdot 2}-\frac{\cdot 09x-\cdot 18}{\cdot 9} \quad \text{.....(1).}$$

$$\frac{7x-4}{x-1}=\frac{7x-26}{x-3} \quad \text{.....(2).}$$

14. A boat's crew rowed $3\frac{1}{2}$ miles down a river and up again in 100 minutes. If the stream had been half as strong again they would have taken $31\frac{1}{4}$ minutes longer; find the rate of the stream.

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15. Simplify $a-[5b-\{a-(3c-3b)+2c-(a-2b-c)\}]$.

16. Multiply $x^5+a^5-ax(x^3+a^3)$ by $x^3+a^3-ax(x+a)$.

17. Find the H.C.F. of

$a^4-pa^3-p^2a^2-p^3a-2p^4$ and $3a^3-7pa^2+3p^2a-2p^3$.

18. Solve the equations:

$$\frac{7x+2}{5}-\frac{4x-1}{2}=0 \quad \text{.....(1).}$$

$$\left.\begin{aligned}\frac{1}{x}-\frac{1}{y}&=\frac{1}{2}\\ \frac{2}{x}+\frac{6}{y}&=5\end{aligned}\right\} \quad \text{.....(2).}$$

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19. Divide $1-\dfrac{y^2}{y^2+a^2}$ by $1+\dfrac{a^2}{y^2+a^2}$.

20. To the sum of $\frac{x}{y}$ and $\frac{y}{x}$ annex ± 2 and multiply the result by xy, putting the product in its simplest form.

21. Solve the simultaneous equations:

$$4x-5y+6z=3,$$
$$8x-7y-3z=9,$$
$$7x-8y+9z=6.$$

22. Find (1) the instant of time between 3 and 4 o'clock at which the hour-hand and minute-hand of a watch are exactly in the same direction, and (2) the instant at which they are exactly opposite each other.

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23. If $a=4$, $b=3$, $c=2$, evaluate

$$\frac{a^2-b^2}{b+c}+\frac{b^2-c^2}{c+a}+\frac{c^2-a^2}{a+b}.$$

24. From $a(a^2+b^2)+b(b^2+c^2)+c(c^2+a^2)$
subtract $ab(b+c)+bc(c+a)+ca(a+b)$.

25. Multiply $a^3+2a^2b+2ab^2+b^3$ by $a^3-2a^2b+2ab^2-b^3$.

26. Find the H.C.F. of
$x^6+2x^5-5x^4-x^2+2x+1$ and $x^5+2x^4-x^3-3x^2+1$.

27. Solve the equations:

$$\frac{x}{2}+\frac{x}{3}=\frac{x}{4}+7 \quad \text{.....(1).}$$

$$\left.\begin{aligned}\frac{x+1}{y-1}-\frac{x-1}{y}&=\frac{6}{y}\\ x-y&=1\end{aligned}\right\} \quad \text{.....(2).}$$

28. Divide 25 into two such parts that the greater may be 49 times the less.

29. If A and B together can do a certain piece of work in 8 days, B and C together in 10 days and C and A together in 9 days, in how many days would each do the work if he worked by himself?

30. There is a number whose three digits are in descending order of magnitude and differ from each other in succession by the same amount. If the number be divided by the sum of its digits, the quotient will be 48; and if from the number 198 be subtracted, the digits of the difference will be the same as those of the original number but in reverse order; find the number.

---

31. If $a=4$, $x=2$, find the numerical value of

$$\frac{2ax^2}{(a-x)^2}-\frac{6\sqrt[3]{ax}}{a^3\sqrt{2a+4x}}-\frac{29x^2}{64a}.$$

32. Divide $a^2x^2-b^2x^2+a^2bx-ab^2x+a^3-a^2b+ab^2-b^3$

by $ax^2+bx^2+abx+a^2+b^2$.

33. Find the L.C.M. of a^2-1, a^3-1 and a^2+5a+4.

34. Reduce to their simplest forms:

$$\left(\frac{a^2}{x}-\frac{x^3}{a^2}\right)\div\left(\frac{a}{x}+\frac{x}{a}\right) \quad \text{.....(1).}$$

$$\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}-\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} \quad \text{.....(2).}$$

35. Shew that the sum of any fraction and its reciprocal is always numerically greater than 2.

36. Solve the equations:

$$6-\frac{x-1}{2}-\frac{x-2}{3}=\frac{3-x}{4} \quad \text{.....(1).}$$

$$\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}=\frac{1}{2} \quad \text{.....(2).}$$

$$\left.\begin{aligned}\frac{5}{x}+\frac{4}{y}&=58\\ \frac{3}{x}+\frac{7}{y}&=67\end{aligned}\right\} \quad \text{.....(3).}$$

37. A vessel can be filled by means of one tap in 3 hours and by means of another tap in 5 hours; in what time will it be filled if both taps run together?

38. A ship left Bombay on a voyage of three weeks with provisions for that time at the rate of a seer a day for each man. At the end of a week a storm arose which washed four men overboard and so damaged the ship that her speed was reduced by half, and each man could be allowed only three-quarters of a seer per diem afterwards; what was the original number of the crew?

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39. If $a=1$, $b=2$, $c=3$, $d=4$, evaluate

$$\frac{ab}{bc}-\frac{bc}{cd}+\frac{ac}{cd}-\frac{cd}{da}+\frac{ad}{da}-\frac{da}{ab}.$$

40. From $(a+b)x+(b+c)y$ take $(a-b)x-(b-c)y$.

41. Multiply $a^4+a^3b+a^2b^2+ab^3+b^4$ by $a-b$.

42. Find the H.C.F. of a^4-x^4 and $a^3-a^2x-ax^2+x^3$.

43. Reduce to their simplest forms:

$$\frac{b-c}{bc}+\frac{c-a}{ca}+\frac{a-b}{ab} \quad \text{.....(1).}$$

$$\frac{a}{b}-\frac{(a^2-b^2)\,x}{b^2}+\frac{a\,(a^2-b^2)\,x^2}{b^2\,(b+ax)} \quad \text{.....(2).}$$

44. If $x+\frac{1}{x}=p$, prove that $x^3+\frac{1}{x^3}=p^3-3p$.

45. Solve the equations:

$$\frac{\sqrt{x+2}}{\sqrt{x-2}}=\frac{3}{2} \quad \text{.....(1).}$$

$$\left.\begin{aligned}\frac{m}{x}+\frac{n}{y}&=a\\ \frac{n}{x}+\frac{m}{y}&=b\end{aligned}\right\} \quad \text{.....(2).}$$

46. Find a number such that, whether divided into two equal parts or into three equal parts, the product of the parts shall be the same.

47. A labourer is engaged for ten days on condition that he shall receive 8 annas for every day's work done and that he shall pay 1 anna for every day on which he is absent from work, and at the end of the ten days he receives 8 annas; how many days did he work and how many was he absent?

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48. Remove the brackets from the expression

$$a-[2b+\{3c-3a-(a+b)\}+2a-(b+3c)].$$

49. If $x=2$, $y=3$, $a=6$, $b=5$, evaluate

$$\sqrt[3]{(x+y)^2}\,b+\sqrt[3]{(x+a)\,(b-2x)}+\sqrt[3]{(b-y)^2}\,x,$$

and

$$x+2a-\{y+b-[x-a-(y-2b)]\}.$$

50. Multiply together $(a^2-3a+2)^2$ and a^2+6a+1.

51. Write down the H.C.F. of

$$(x-1)^2\,(x-2)\,(x-3) \text{ and } (x-1)^3\,(x-4)\,(x-5).$$

52. Multiply $\frac{x^2}{a^2}-\frac{x}{a}+1$ by $\frac{x^2}{a^2}+\frac{x}{a}+1$.

53. Simplify $\frac{1}{a\,(a-b)\,(a-c)}+\frac{1}{b\,(b-c)\,(b-a)}+\frac{1}{c\,(c-a)\,(c-b)}$.

54. Shew that

$$\frac{a\sqrt{a+x}}{\sqrt{a+x}-\sqrt{x}}=a+x+\sqrt{ax+x^2}.$$

55. Solve the equations:

$$\frac{2x-3}{3x-4}=\frac{4x-5}{6x-7} \quad \text{.....(1).}$$

$$\sqrt{x+48}+\sqrt{x}=12 \quad \text{.....(2).}$$

$$\sqrt[m]{a+x}=\sqrt[2m]{x^2+8ax+b^2} \quad \text{.....(3).}$$

56. A cistern can be filled by two pipes A and B in 12 hours and by the pipe A alone in 20 hours; find the time in which it would be filled by B alone.

57. A privateer sailing at the rate of 10 miles an hour discovers a ship 18 miles off running from her at the rate of 8 miles an hour; how many miles can the ship run before she is overtaken?

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58. If $a=16$, $b=10$, $x=5$, $y=1$, find the numerical value of

$$(a-y)\{\sqrt{(2bx)}+x^2\}+\sqrt{\{(a-x)(b+y)\}}.$$

59. Divide $6a^4-a^3b+2a^2b^2+13ab^3+4b^4$ by $2a^2-3ab+4b^2$.

60. Prove that

$$\{(b-c)^2+(c-a)^2+(a-b)^2\}^2=2\{(b-c)^4+(c-a)^4+(a-b)^4\}.$$

61. Simplify $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$.

62. Extract the square root of

$$4x^4-4x^3+5x^2-2x+1 \quad \text{.....(1).}$$

$$x^4-x^3+\frac{x^2}{4}+4x-2+\frac{4}{x^2} \quad \text{.....(2).}$$

63. A lb. of tea and three lbs. of sugar cost six shillings, but if sugar were to rise 50 per cent. and tea 10 per cent. they would cost seven shillings; find the price of tea and sugar respectively.

64. A railway train after travelling for an hour meets with an accident which delays it an hour, after which it proceeds at three-fifths of its former rate and arrives at the terminus three hours behind time; had the accident happened 50 miles further on, the train would have arrived 1 hour 20 minutes sooner. Find the length of the line.

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65. If $a=3$, $b=-2$, $c=1$, evaluate

$$\frac{a^2+b^2+c^2}{a+b+c}-\frac{a^2-b^2-c^2}{a-b-c} \quad \text{.....(1).}$$

$$a(a-b)(a-c)\{\sqrt{2a^2+3b+4c}-\sqrt{(a+b)(a+c)}\} \quad \text{.....(2).}$$

66. Extract the square root of

$$x^6-2x^4y^2+2x^3y^3+x^2y^4-2xy^5+y^6.$$

67. Add together

$$\frac{1}{a+b}, \quad \frac{1}{a-b}, \quad \frac{2b}{a^2-b^2}.$$

68. Shew that

$$\frac{x+y}{\sqrt{x+y}-y} = \frac{\sqrt{x^3+x^2y-xy^2-y^3}}{\sqrt{x^2-y^2}-y\sqrt{x-y}}.$$

69. Solve the equations:

$$\sqrt{x+12}+\sqrt{x}=6 \quad \ldots\ldots(1).$$

$$\frac{2x-3}{5x-2}=\frac{2x+6}{5x+37} \quad \ldots\ldots(2).$$

$$\left.\begin{aligned}\frac{2}{x}+\frac{5}{y}&=\frac{5}{6}\\ \frac{3}{x}+\frac{4}{y}&=\frac{9}{10}\end{aligned}\right\} \quad \ldots\ldots(3).$$

70. Find the exact time between 3 and 4 o'clock when the hour-hand and minute-hand of a watch are at right angles to each other.

71. A merchant has a share in the Back Bay Company and a share in the Mazagon Company; the market rate for the two shares was Rs. 2000, but Mazagon shares rose 10 per cent. and Back Bays fell 20 per cent., in consequence of which the value of the two shares became $12\frac{1}{2}$ per cent. less than before. Find the original market value of each share.

72. Two boats start at the same time from Bassein and Tanna, the distance between which is 18 miles; at a distance of one mile from Tanna the Callian creek falls into the Tanna creek, causing a current at the rate of $2\frac{1}{2}$ miles an hour towards Tanna and of 2 miles an hour towards Bassein. The boat from Tanna is rowed at the rate of $3\frac{1}{3}$ miles an hour and the Bassein boat at 3 miles an hour; where will they meet?

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73. Substitute $y+3$ for x in the expression

$$x^4-x^3+2x^2-3.$$

74. Find the L.C.M. of

x^2-4a^2, $x^3+2ax^2+4a^2x+8a^3$ and $x^3-2ax^2+4a^2x-8a^3$.

75. Simplify

$$\frac{5}{2(x+1)}-\frac{1}{10(x-1)}-\frac{24}{5(2x+3)} \quad \ldots\ldots(1).$$

$$\left(\frac{a}{a+b}+\frac{b}{a-b}\right)\div\left(\frac{a}{a-b}-\frac{b}{a+b}\right) \quad \ldots\ldots(2).$$

76. Solve the equations:

$$\frac{x}{4}-\frac{5x+8}{6}=\frac{2x-9}{3} \quad \ldots\ldots(1).$$

$$(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2 \quad \ldots\ldots(2).$$

$$\left.\begin{aligned} ax+by&=c\\ mx-ny&=d\end{aligned}\right\} \quad \ldots\ldots(3).$$

$$\left.\begin{aligned} \frac{x}{a}+\frac{y}{b}&=1\\ \frac{x}{3a}+\frac{y}{6b}&=\frac{2}{3}\end{aligned}\right\} \quad \ldots\ldots(4).$$

77. Extract the square root of

$$4x^4-12ax^3+25a^2x^2-24a^3x+16a^4.$$

78. Two shepherds who own a flock of sheep agree to divide its value; one of them takes 72 sheep and the other takes 92 sheep and pays the first Rs. 350; what is the value of a sheep?

79. A and B start simultaneously from Poona to go to Kirkee. A would reach Kirkee half an hour before B but, missing his way, goes a mile and back again needlessly, during which he walks at twice his former pace, and he reaches Kirkee 6 minutes before B; C starts twenty minutes after A and B and walking at the rate of $2\frac{1}{7}$ miles an hour arrives at Kirkee ten minutes after B. Find the rates of walking of A and B and the distance from Poona to Kirkee.

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80. Reduce to its simplest form

$$\{2x^2-(y^2-xy)\}-\{y^2-(4xy+x^2)\}+\{2y^2-(3xy-x^2)\}.$$

81. Divide $a^5-2a^4b+2a^3b^2-4a^2b^3-8ab^4+16b^5$ by a^2-2b^2.

82. Resolve into two factors

$$a^2-4b^2-c^2+9d^2-2(3ad-2bc).$$

83. Find the H.C.F. of

$4a^5x+6a^4x^2-18a^3x^3$ and $12a^5x^2-34a^4x^3+28a^3x^4-6a^2x^5$.

84. Find the value of

$$\left(\frac{a+b}{a-b}-\frac{a-b}{a+b}-\frac{2a^2}{a^2-b^2}\right)\frac{a-b}{2a}.$$

85. Extract the square root of

$$x^6-4x^5y+4x^4y^2-2x^3y^2+4x^2y^3+y^4.$$

86. Solve the equations:

$$\frac{4x-9}{27}-\frac{x-3}{4}=\frac{5x-3}{6}-\frac{x+6}{2} \quad \ldots\ldots(1).$$

$$\frac{75-x}{3(x+1)}+\frac{80x+21}{5(3x+2)}=\frac{23}{x+1}+5 \quad \ldots\ldots(2).$$

$$\left.\begin{aligned}\frac{9x-2y}{30}-\frac{3x-2y}{36}&=\frac{4}{9}\\ \frac{5x-4y}{60}+1\tfrac{1}{10}&=2x-2\tfrac{2}{3}\end{aligned}\right\} \quad \ldots\ldots(3).$$

87. The charge for first-class tickets of admission to an exhibition was Rs. 4 each and for second-class tickets Rs. 2. 8 as.; the whole number of tickets sold was 259 and the total amount received for them Rs. 731. 8 as. How many first-class tickets were sold, and how many second-class tickets?

88. *A* and *B* agreed to reap a field for Rs. 20; if they had worked together every day the field would have been reaped in 15 days, but at the end of 7 days *A* left off working for 4 days and it consequently took $16\frac{1}{2}$ days to reap the field. In how many days could *A* alone, and in how many days could *B* alone, have reaped the field? And what share of the Rs. 20 ought each to receive for the work he actually did?

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89. Find the L.C.M. of

$2x^4+2x^3y-18x^2y^2+22xy^3-8y^4$ and $3x^4-3x^3y-9x^2y^2+15xy^3-6y^4$.

90. Reduce to its simplest form

$$\left(\frac{x^2}{yz}+\frac{x}{z}-\frac{x+y}{u}\right)\div\left(\frac{x}{y}-\frac{z}{u}\right).$$

91. If $2s=a+b+c$, prove that

$$s(s-b)(s-c)+s(s-c)(s-a)+s(s-a)(s-b)=(s-a)(s-b)(s-c)+abc.$$

92. Solve the equations:

$$\frac{10x+4}{21}+\frac{7-2x^2}{14(x-1)}=\frac{11-5x}{15}+\frac{4x-3\frac{2}{5}}{6} \quad \ldots\ldots(1).$$

$$\left.\begin{aligned}10x-\frac{4y+3}{4}&=7+\frac{12y-10x}{5}\\ 16y+\frac{5x-2}{3}&=26\tfrac{1}{2}-\frac{8y+1}{2}\end{aligned}\right\} \quad \ldots\ldots(2).$$

93. A garrison of 1500 men was victualled for 36 days, but after 16 days it was reinforced, and the provisions were then exhausted in 12 days; find the number of men in the reinforcement.

94. A person left Poona in the Satara mail tonga at 2 P.M., and having proceeded a certain distance he alighted and returned to Poona on foot, walking at the rate of 3 miles an hour, and he reached Poona at 8 P.M.; if he had gone six miles further in the tonga he would not have reached Poona till 10.40 P.M. How far did he go towards Satara and what was the speed of the tonga?

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95. Find the value of $\dfrac{x^4-a^4}{x^2-2ax+a^2} \div \dfrac{x^2+ax}{x-a}$.

96. Simplify $\dfrac{1}{4a^3(a+x)} + \dfrac{1}{4a^3(a-x)} + \dfrac{1}{2a^2(a^2+x^2)}$.

97. Find the H.C.F. of $6x^3+13x^2+6x$ and $8x^3+6x^2-9x$.

98. Reduce to its lowest terms the fraction

$$\frac{x^4+7x^3+7x^2-15x}{x^3-2x^2-13x+110}.$$

99. Extract the square root of

$$a^4-2a^3+\tfrac{3}{2}a^2-\tfrac{1}{2}a+\tfrac{1}{16}.$$

100. Solve the simultaneous equations:

$$u+2x+y+2z=-3,$$
$$2u+x+2y+z=3,$$
$$u+2x+12y+z=21,$$
$$u+x+6y+z=10.$$

101. A garrison had sufficient provisions for 30 months, but at the end of 4 months the number of troops was doubled, and 3 months after it was reinforced with 400 more men, and the provisions lasted only 15 months altogether; how many men formed the garrison originally?

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102. Divide the product of $y^3-12y+16$ and $y^3-12y-16$ by y^2-16, and $(x^2-yz)^3+8y^3z^3$ by x^2+yz.

103. Resolve $x^{16}-y^{16}$ into five factors.

104. Find the H.C.F. of $2x^5-11x^2-9$ and $4x^5+11x^4+81$.

105. Solve the equations:

$$\frac{2x-1}{3}-\frac{3x-2}{4}=\frac{5x-4}{6}-\frac{7x+6}{12} \quad \ldots\ldots\ldots\ldots (1).$$

$$\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7} \quad \ldots\ldots\ldots\ldots (2).$$

$$(x+\tfrac{3}{2})(x-\tfrac{5}{2})=(x+3)(x-5)+\tfrac{93}{4} \quad \ldots\ldots\ldots\ldots (3).$$

$$(x-\tfrac{3}{2})(x+\tfrac{5}{2})=(x-3)(x+5)+\tfrac{3}{4} \quad \ldots\ldots\ldots\ldots (4).$$

106. Divide Rs. 15,200 among Krishna, Gopál and Govind, so that Gopál shall have Rs. 1,000 more than Krishna and Govind Rs. 2,700 more than Gopál.

107. A certain sum is to be divided among A, B and C; A is to have Rs. 300 less than half, B Rs. 100 less than a third and C Rs. 80 more than a quarter; how much will each receive?

108. One-tenth of a rod is coloured red, one-twentieth orange, one-thirtieth yellow, one-fortieth green, one-fiftieth blue, one-sixtieth indigo, and the remainder, which is 302 inches long, violet; find the length of the rod.

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109. Divide $a^2+1+\frac{1}{a^2}$ by $a-1+\frac{1}{a}$, and

$$\frac{x^2}{2y^2}-4+\frac{6y^2}{x^2} \text{ by } \frac{x}{2y}-\frac{3y}{x}.$$

110. Find the H.C.F. of x^3+2x^2+2x+1 and x^3-2x-1; and the L.C.M. of $6(x^2+xy)$, $8(xy-y^2)$ and $10(x^2-y^2)$.

111. Extract the square root of

$$4x^4+12x^3y+13x^2y^2+6xy^3+y^4.$$

112. Solve the equations:

$$\tfrac{1}{3}(x+1)+\tfrac{1}{4}(x+3)=\tfrac{1}{5}(x+4)+16 \quad \text{.....(1).}$$

$$\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5} \quad \text{.....(2).}$$

$$\left.\begin{aligned}\frac{x+1}{2}+3y&=33\\ \frac{y+6}{3}+2x&=27\end{aligned}\right\} \quad \text{.....(3).}$$

113. There is a number consisting of two digits whose sum is 10, and, if 72 be subtracted from it, the digits will be reversed; what is the number?

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114. Simplify

$$\frac{x^4-y^4}{a^2b+ab^2}\times\frac{a+b}{(x+y)^2}\div\frac{(x-y)^2}{ab}.$$

115. Find the square root of

$$x^6-4x^5+10x^4-20x^3+25x^2-24x+16;$$

and the cube root of

$$8x^3+60x^2y+150xy^2+125y^3.$$

116. Solve the equations:

$$(3x-1)^2+(4x-2)^2=(5x-3)^2 \quad \text{.....(1).}$$

$$\left.\begin{aligned}4x^2+2xy-6y^2+288&=(2x+3y-131)(2x-2y+13)\\ 5x-4y&=22\end{aligned}\right\} \quad \text{.....(2).}$$

$$\left.\begin{aligned}\frac{2}{x}-\frac{3}{y}+\frac{4}{z}&=\frac{33}{10}\\ \frac{1}{x}+\frac{1}{y}-\frac{1}{z}&=-\frac{37}{60}\\ \frac{5}{x}-\frac{2}{y}+\frac{3}{z}&=\frac{191}{60}\end{aligned}\right\} \quad \text{.....(3).}$$

117. B has 5 miles start of A but only travels at the rate of 3 miles an hour, while A travels at the rate of $4\frac{1}{2}$ miles an hour; where will A overtake B and how long will he take to do it?

118. A mixture is made up of a gallons of one liquid at m rupees per gall ɔ, b gallons of a second liquid at n rupees per gallon and c gallons of a third liquid at p rupees per gallon; what will be the value of the mixture?

119. The sum of the three digits of which a number consists is 9; the first digit is one-eighth of the number consisting of the last two, and the last digit is likewise one-eighth of the number consisting of the first two; find the number.

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120. Reduce to its lowest terms the fraction

$$\frac{3a^3-3a^2b+ab^2-b^3}{4a^2-ab-3b^2}.$$

121. Divide $\frac{a}{1+a}+\frac{1-a}{a}$ by $\frac{a}{1+a}-\frac{1-a}{a}$.

122. Extract the square root of

$$a^2-2ax+x^2+2a-2x+1.$$

123. Solve the simultaneous equations:

$$x+y-z=3,$$
$$y+z-x=7,$$
$$z+x-y=5.$$

124. A man has a number of rupees which he tries to arrange in the form of a square; on the first attempt he has 116 over, and when he increases the side of the square by three rupees he wants 25 to complete the square; how many rupees has he?

125. A fraction becomes equal to 3 when 13 is added to its numerator and becomes equal to 1 when 1 is subtracted from its denominator; find the fraction.

126. A person walked to the top of a mountain at the uniform rate of $2\frac{1}{3}$ miles an hour and down again by the same way at the rate of $3\frac{1}{2}$ miles an hour, and he was 5 hours in going and returning; how many miles did he walk?

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127. Multiply $x^2+y^2+z^2+1$ by $x^2+y^2-z^2-1$, and divide $243x^5-y^5$ by $3x-y$.

128. Multiply $\frac{a^2+ab}{a^2+b^2}$ by $\frac{a}{a-b}$ and by $\frac{b}{a+b}$, and find the difference of the results.

129. Reduce to its simplest form

$$\left(\frac{2x^3-2y^3}{x-y}-\frac{x^4-y^4}{x^2-y^2}\right)\div\left(\frac{x^3-xy^2}{x-y}-\frac{x^4-y^4}{x^2+y^2}\right).$$

130. Find the H.C.F. of $x^3-3a^2x-2a^3$ and $x^3-ax^2-4a^3$, and the L.C.M. of x^2-4, x^2-5x+6 and x^2-9.

131. Extract the square root of

$$4x^4+6x^3+\tfrac{89}{4}x^2+15x+25;$$

and the cube root of

$$x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right).$$

132. Solve the equations:

$$1+2\sqrt{x}=\sqrt{(4x+\sqrt{16x+2})} \quad \ldots\ldots(1).$$

$$\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6} \quad \ldots\ldots(2).$$

$$\left.\begin{aligned} 5z-4y&=4+3x\\ 2x+4z&=8+3y\\ 2y+x&=14-3z \end{aligned}\right\} \quad \ldots\ldots(3).$$

133. Divide the number 99 into five such parts that the first may exceed the second by 3, be less than the third by 10, greater than the fourth by 9 and less than the fifth by 16.

134. A person being asked what o'clock it was answered that it was between 5 and 6 and that the hour and minute hands were together; what was the time?

135. A quantity of grain was divided equally among a certain number of poor persons; if there had been ten more each would have received one seer less than he did, and if there had been eight fewer each would have received one seer more than he did; find the number of persons and the quantity of grain each received.

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136. Find the value of

$$\frac{\sqrt{3-2x^2}-x}{x(1+3x)-x^3}, \text{ when } x=-\tfrac{1}{3};$$

and of

$$1-[1-\{1-(1-\overline{1-1})\}].$$

137. Simplify

$$\frac{9a^2b^2}{16(x+y)}\div\left[\frac{3a(x-y)}{7(c+d)}\div\left\{\frac{4(c-d)}{21ab^2}\div\frac{c^2-d^2}{4(x^2-y^2)}\right\}\right] \quad \ldots\ldots(1).$$

$$\left(\sqrt{\frac{a+x}{x}}-\sqrt{\frac{x}{a+x}}\right)^2-\left(\sqrt{\frac{x}{a}}-\sqrt{\frac{a}{x}}\right)^2+\frac{x^2}{a(a+x)} \quad \ldots(2).$$

138. Find the coefficient of x in the quotient obtained by dividing $8x^4+xy^3-y^4$ by $x-\frac{y}{2}$.

139. Separate into their simplest factors

$$x^2-xy-6y^2 \text{ and } x^3-4xy^2-x^2y+4y^3.$$

140. If $xy=ab(a+b)$ and $x^2-xy+y^2=a^3+b^3$, shew that

$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{b}-\frac{y}{a}\right)=0.$$

141. Find by inspection the L.C.M. of $2x^2-8$, $3x^2-9x+6$ and $6x^2+18x+12$.

142. Extract the square root of

$$4x^4-12x+\frac{25}{x^2}-\frac{24}{x^5}+\frac{16}{x^8};$$

and the cube root of

$$8x^9-12x^8+6x^7-37x^6+36x^5-9x^4+54x^3-27x^2-27.$$

143. What value of c makes

$$(x-2)^2-(x-1)(x-3)=c$$

an identity? Can any value of c make it an equation?

144. If the telegraph posts by the side of a railway be 60 yards apart, shew that twice the number passed by a train in a minute gives roughly the number of miles per hour at which the train is moving. If eleven posts be passed in a minute, in what time would the distance traversed, estimated by this rule, be one mile in error?

145. A boy receives a fixed sum as pocket-money at the beginning of every week, and in each week he spends half of all that he had at its beginning; he had no money before the first pocket-money was given him and at the end of the third week he has 1*s.* 2*d.*; what is his weekly allowance?

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146. Remove the brackets from the expression

$$\{m-n-(3x-2y)\}-\{2x+5y-(m+n)\}.$$

147. Find the quotient which arises from dividing the third power of $10a^2$ by the square root of one million times a^{12}.

148. Find the H.C.F. of

$$21x^3-26x^2+8x \text{ and } 6x^2-x-2;$$

and the L.C.M. of

$$x^2-1,\ x^2+2x-3 \text{ and } x^3-7x^2+6x.$$

149. Reduce to its simplest form

$$\left(a+\frac{b-a}{1+ba}\right)\div\left(1-a\frac{b-a}{1+ba}\right).$$

150. Extract the square root of

$$x^4-6x^3y+13x^2y^2-12xy^3+4y^4.$$

151. Solve the simultaneous equations:

$$\frac{4}{x}-\frac{5}{y}=\frac{x+y}{xy}+1\tfrac{2}{35},$$
$$\frac{1}{x}-\frac{1}{y}=\frac{34}{35}.$$

152. From a certain sum of money I took away one-third part and put in its stead 50 rupees; from the sum thus increased I took away one-fourth part and put in its stead 70 rupees, and I then found I had 120 rupees; what was the original sum?

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153. Reduce to the simplest form

$$a^2+2d^2-(2e^2-b^2)-\{(d^2-c^2-e^2)+(d^2-e^2)\}.$$

154. Find the H.C.F. of

$$2x^2-xy-6y^2 \text{ and } 3x^2-8xy+4y^2.$$

155. Add together $\frac{x(x+3)}{(x+1)(x+2)}$ and $\frac{2}{3x(x+2)}$, and find the value of the sum when $x=\frac{1}{2}$.

156. Extract the square root of

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{x}{y}-\frac{y}{x}+\frac{9}{4}.$$

157. A and B invested equal sums; A gained Rs. 1000 and B lost so much that his money became two-thirds of A's money. If each were to give the other a third of his present money, B's loss would be diminished by a half; how much did each invest?

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158. Find the continued product of

$$a+b+c,\quad -a+b+c,\quad a-b+c,\quad a+b-c.$$

159. From the sum of the squares of $\frac{1}{a-b}$ and $\frac{1}{a+b}$ subtract the square of $\frac{2b}{a^2-b^2}$.

160. Find the H.C.F. of the expressions forming the numerator and denominator of the fraction $\frac{x^4-15x^2+28x-12}{2x^3-15x+14}$, and reduce it to its lowest terms.

161. Solve the simultaneous equations:

$$x+y+z=6,$$
$$3x-y+2z=7,$$
$$4x+3y-z=7.$$

162. Determine the time between ten and twelve o'clock at which the hour and minute hands of a clock are exactly together.

163. One student said to another: "If you give me half your money, I shall have a hundred rupees." The other replied: "I shall have a hundred rupees if you give me a third of your money." How much had each?

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164. Divide $a^2x^8+(2ac-b^2)\,x^4+c^2$ by ax^4-bx^2+c.

165. Resolve the expression

$$2\,(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4)\cdot$$

into four factors.

166. Find the H.C.F. and L.C.M. of a^4-x^4 and $a^3-a^2x-ax^2+x^3$.

167. Solve the equations:

$$\frac{2x-9}{27}+\frac{x}{18}-\frac{x-3}{4}=8\tfrac{1}{3}-x \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$1{\cdot}2x-\frac{{\cdot}18x-{\cdot}05}{{\cdot}5}={\cdot}4x+8{\cdot}9 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$$\left.\begin{aligned}\frac{x+6}{y}&=\frac{3}{4}\\ \frac{x}{y-2}&=\frac{1}{2}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

168. A train carrying three classes of passengers at 6 as., 4 as. and 3 as., has eight times as many third class passengers as there are of the second class and seven times as many second class passengers as there are of the first class, and the whole sum received is Rs. 290. 6 as. How many first class passengers are there?

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169. If $\left(a+\frac{1}{a}\right)^2=3$, prove that $a^3+\frac{1}{a^3}=0$.

170. Resolve into their simplest factors

(1) $6x^2+5x-6$, (2) $3x^2-10x-8$, (3) $9x^4-82x^2y^2+9y^4$.

171. Find the H.C.F. of $x^4-3x+20$ and $5x^4-3x^3+64$.

172. Extract the cube root of

$$x^3+\frac{8}{x^3}-12\left(x^2+\frac{4}{x^2}\right)+54\left(x+\frac{2}{x}\right)-112.$$

173. Solve the equations:

$$\tfrac{1}{5}(7x+1)-\tfrac{1}{3}(17-2x)=\tfrac{1}{4}(5x+1) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\frac{x}{a+b}+\frac{x-2a}{b-a}=2\,\frac{x+2a}{3a+b} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

174. A father's age is four times that of his elder son and five times that of his younger son; when the elder son has lived to three times his present age, the father's age will exceed twice that of his younger son by three years; find their present ages.

175. A cottage cost Rs. 1500 to build; at what rent must it be let to pay five per cent. clear after allowing ten per cent. of the receipts for repairs?

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176. If $x+y=2z$, shew that

$$\frac{x}{x-z}+\frac{y}{y-z}=2.$$

177. Divide $x^3+8y^3-27z^3+18xyz$ by $x+2y-3z$.

178. Resolve into factors of the first degree the expression

$$a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc.$$

179. If $x=\dfrac{4ab}{a+b}$, find the value of

$$\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}.$$

180. Solve the equations:

$$\frac{17-3x}{5}-\frac{2+4x}{3}=\frac{14+7x}{3}+5-6x \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$(x-a)(x-b)=(x+a)(x+b) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

181. A criminal having escaped from prison travelled for ten hours before his escape was known, and was then pursued so as to be gained upon three miles an hour. After his pursuers had travelled for three hours they met an express going at the same rate as themselves which had met the criminal 2 hours 24 minutes before; in what time after the commencement of the pursuit will they overtake him?

182. Divide the number 127 into four such parts that the first increased by 18, the second diminished by 5, the third multiplied by 6, and the fourth divided by $2\frac{1}{2}$, shall all be equal.

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183. Divide $(4x^3-3a^2x)^2+(4y^3-3a^2y)^2-a^6$ by $x^2+y^2-a^2$.

184. If

$$(a+b)(b+c)(c+d)(d+a)=(a+b+c+d)(bcd+cda+dab+abc),$$

prove that $ac=bd$.

185. Resolve $10x^2-23x-5$ into its simplest factors.

186. Shew that

$$2(a-b)(a-c)+2(b-c)(b-a)+2(c-a)(c-b)$$

is the sum of three squares.

187. Find the H.C.F. of

$7x^4-10ax^3+3a^2x^2-4a^3x+4a^4$ and $8x^4-13ax^3+5a^2x^2-3a^3x+3a^4$.

188. A certain number consisting of two digits becomes 110 when the number obtained by reversing the digits is added to it; also the original number exceeds unity by five times the excess of the second number over unity; what is the number?

189. A person walks from A to B, a distance of $7\frac{1}{2}$ miles, in 2 hours $17\frac{1}{2}$ minutes, and returns in 2 hours 20 minutes, his rates of walking up hill, down hill and on the level being 3, $3\frac{1}{2}$ and $3\frac{1}{4}$ miles an hour respectively; find the length of level road between A and B.

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190. Find the continued product of

x^2-2y^2, $x^2-2xy+2y^2$, x^2+2y^2 and $x^2+2xy+2y^2$.

191. Resolve into factors and thence find the L.C.M. of

$a^2+6ab+5b^2$, $a^3-a^2b-ab^2+b^3$ and $a^2+5ab-6b^2$.

192. Simplify the following fractional expressions:

$$\left(x+\frac{16x-27}{x^2-16}\right)\div\left(x-1+\frac{13}{x+4}\right) \quad \text{.....(1).}$$

$$\frac{\sqrt{(a^4-4a^3+12a^2-16a+16)}}{\sqrt[3]{(a^9+24a^6+192a^3+512)}} \quad \text{.....(2).}$$

193. Solve the equations:

$$\frac{2x}{3}-\frac{1-\frac{x}{2}}{4x}=\frac{2x+7}{12}-\frac{1-x}{2} \quad \text{.....(1).}$$

$$\left.\begin{aligned}\frac{p}{x}+\frac{q}{y}&=0\\ px+qy&=r\end{aligned}\right\} \quad \text{.....(2).}$$

194. I bought a horse and carriage for £90; I sold the horse at a gain of 12 per cent. and the carriage at a loss of 4 per cent. and gained on the whole 6 per cent.; what was the original cost of the carriage?

195. A man walks one-third of the distance from A to B at the rate of a miles an hour and the remainder at the rate of $2b$ miles an hour, and travelling back from B to A at the rate of $3c$ miles an hour takes the same time; prove that

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{c}.$$

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196. Simplify

$$24\{x-\tfrac{1}{2}(x-1)\}\{x-\tfrac{2}{3}(x-2)\}\{x-\tfrac{3}{4}(x-1\tfrac{1}{3})\};$$

and subtract the result from

$$\frac{(x^2+7x+12)(x^2-x-6)}{x-3}.$$

197. Reduce to its simplest form

$$\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}.$$

198. Resolve into factors the expression

$$4(ad-bc)^2-(a^2+d^2-b^2-c^2)^2.$$

199. Find the H.C.F. and L.C.M. of

$x^5+x^4-4x^3+2x^2+6x-9$, x^4-x^2+6x-9 and $x^4+2x^3-5x^2-6x+9$.

200. Extract the square root of

$$\frac{x^6}{16}-\frac{x^4}{6}-\frac{x^3}{4}+\frac{x^2}{9}+\frac{x}{3}+\frac{1}{4};$$

and the cube root of

$$8x^6-36x^5+66x^4-63x^3+33x^2-9x+1.$$

201. Solve the equations:

$$\sqrt{(x-a)^2+2ab+b^2}=x-a+b \quad \ldots\ldots(1).$$

$$\left.\begin{aligned}\frac{15}{x}-\frac{1}{y}&=4\tfrac{1}{2}\\ \frac{9}{x}+\frac{2}{y}&=4\end{aligned}\right\} \quad \ldots\ldots(2).$$

202. A man walked from the University Library to Malabar Hill at the rate of 3 miles an hour, ran part of the way back at the rate of $8\frac{1}{2}$ miles an hour, and then walked the remainder in 1 hour 5 minutes. He was out altogether 2 hours 44 minutes; find the distance between Malabar Hill and the University.

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203. If $a+b=c+d$, prove that each of these quantities is equal to

$$\frac{abcd}{ab+cd}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

204. If $x+\frac{1}{y}=y+\frac{1}{z}=1$, prove that $z+\frac{1}{x}=1$ and $xyz+1=0$.

205. Simplify

$$\frac{(b+c)(x^2+a^2)}{(c-a)(a-b)}+\frac{(c+a)(x^2+b^2)}{(a-b)(b-c)}+\frac{(a+b)(x^2+c^2)}{(b-c)(c-a)} \quad \ldots\ldots(1).$$

$$\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)-\left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right) \quad \ldots\ldots(2).$$

206. Shew by means of a formula that

$$(ax+by+cz)^3+(cx-by+az)^3$$

is divisible by $(a+c)(x+z)$.

207. Resolve into three factors the expression

$$x^3-2x^2-23x+60.$$

208. Extract the square root of

$$(a-b)^2\{(a-b)^2-2(a^2+b^2)\}+2(a^4+b^4).$$

209. Solve the equations:

$$\frac{x-8}{x-10}-\frac{x-5}{x-7}=\frac{x-7}{x-9}-\frac{x-4}{x-6} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\frac{x-bc}{a}-\frac{x-a(a+b)}{c}=\frac{cx}{(a+b)(a+c)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

210. A number consists of three digits, the right-hand one being zero; if the left-hand and middle digits be interchanged, the number is diminished by 180; if the left-hand digit be halved, and the middle and right-hand digits be interchanged, the number is diminished by 336; find the number.

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211. Resolve into factors

$$x^4-(p^2+2)x^2y^2+y^4.$$

212. Given the relation

$$\frac{1-2bx+b^2}{1-b^2}=\frac{1-b^2}{1+2by+b^2};$$

prove that

$$\frac{x-y}{1-xy}=\frac{2b}{1+b^2}.$$

213. Divide

$$1+a+a^2+a^3+a^4+a^6+a^7+a^8+a^9+a^{15} \text{ by } 1+a^5+a^6.$$

214. Simplify the fraction

$$\frac{(y-z)(y+z)^3+(z-x)(z+x)^3+(x-y)(x+y)^3}{(y+z)(y-z)^3+(z+x)(z-x)^3+(x+y)(x-y)^3}.$$

215. Solve the simultaneous equations:

$$\frac{a-b}{x}+\frac{a+b}{y}=2\frac{a^2+b^2}{a^2-b^2},$$

$$\frac{a+b}{x}+\frac{a-b}{y}=2.$$

216. A number consists of three digits whose sum is 10; the middle digit is equal to the sum of the other two, and the number will be increased by 99 if its digits are reversed; what is the number?

217. If 19 lbs. of gold weigh 18 lbs. in water and 10 lbs. of silver weigh 9 lbs. in water; find the quantity of gold and of silver in a mass of gold and silver weighing 106 lbs. in air and 99 lbs. in water.

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218. Find the divisor when $(4x^2+7xy+5y^2)^2$ is the dividend, $8(x+2y)^2$ the quotient, and $y^2(9x+11y)^2$ the remainder.

219. Find in terms of a the value of the expression

$$x(y+2)+\frac{x}{y}+\frac{y}{x},$$

when $x=\dfrac{y}{y+1}$ and $y=\dfrac{a-2}{2}$.

220. Simplify $$\frac{\left(p+\frac{1}{q}\right)^m\left(p-\frac{1}{q}\right)^n}{\left(q+\frac{1}{p}\right)^m\left(q-\frac{1}{p}\right)^n}.$$

221. If $a+b=1$, prove that

$$(a^2-b^2)^2=a^3+b^3-ab.$$

222. Solve the equations:

$$\frac{a+x}{a^2+ax+x^2}+\frac{a-x}{a^2-ax+x^2}=\frac{2a^4}{x(a^4+a^2x^2+x^4)} \quad \ldots\ldots\ldots(1).$$

$$\frac{2x+3}{x+1}=\frac{4x+5}{4x+4}+\frac{3x+3}{3x+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

223. Shew that if a number of two digits is four times the sum of its digits, the number formed by interchanging the digits is seven times their sum.

224. A certain resolution was carried in a debating society by a majority which was equal to one third of the number of votes given on the losing side; but if with the same number of votes 10 more votes had been given to the losing side, the resolution would have been carried by a majority of one only. Find the number of votes given on each side.

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225. If $x=1$, $y=\frac{1}{2}$, $z=0$, find the value of

$$[x-\{y-z-(2x-2y-\tfrac{1}{2}\overline{3z-y})\}]\div\left\{x-y\div\left(z-\frac{1}{x}\right)\right\}.$$

226. Find the factors of

(1) $8x^3+729y^9$, (2) $x^5+7x^3-5x^2-35$, (3) $(x^2+4x)^2-2(x^2+4x)-15$.

227. Multiply $x^2+y^2+z^2-yz-zx-xy$ by $x+y+z$, and from the result shew that

$$(y-z)^3+(z-x)^3+(x-y)^3-3(y-z)(z-x)(x-y)=0.$$

228. Find the expression of lowest dimensions which is exactly divisible by $a^2b-b(b-c)^2$, $ac^2-a(a-b)^2$ and $(a+c)^2c-b^2c$.

229. Extract the square root of

$$(2x+1)(2x+3)(2x+5)(2x+7)+16.$$

230. Simplify

$$\frac{\frac{a}{b}+\frac{b}{a}+2}{a+b}+\frac{\frac{a}{b}+\frac{b}{a}-2}{a-b} \quad \ldots\ldots(1).$$

$$\left(\frac{x^6+y^6}{x^6-y^6}\times\frac{x-y}{x+y}\right)\div\frac{x^4-x^2y^2+y^4}{x^4+x^2y^2+y^4} \quad \ldots\ldots(2).$$

231. Solve the equations:

$$2x-[3-\{4x+(x-1)\}-5]=8 \quad \ldots\ldots(1).$$

$$\left.\begin{aligned}2^x&=8^{y+1}\\9^y&=3^{x-9}\end{aligned}\right\} \quad \ldots\ldots(2).$$

$$\left.\begin{aligned}\frac{1}{1-x+y}-\frac{1}{x+y-1}&=\frac{2}{3}\\\frac{1}{1-x+y}-\frac{1}{1-x-y}&=\frac{4}{3}\end{aligned}\right\} \quad \ldots\ldots(3).$$

232. A man allowed a pauper 3*s.* 6*d.* per week until the number of shillings left was the same as the number of weeks he had been paying; he then increased the allowance to 4*s.* per week; if the money lasted altogether for 35 weeks, find how much the man gave away.

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233. Find the value of

$$\frac{x^3-3abx-2b^3}{x^2-ab}+\frac{x^2-4ab}{x-2a},$$

when $x=a+b$.

234. If $m=a^x$, $n=a^y$, $a^2=(m^yn^x)^z$; shew that $xyz=1$.

235. Simplify

$$\frac{9y^2-(4z-2x)^2}{(2x+3y)^2-16z^2}+\frac{16z^2-(2x-3y)^2}{(3y+4z)^2-4x^2}+\frac{4x^2-(3y-4z)^2}{(4z+2x)^2-9y^2}.$$

236. Find the H.C.F. and L.C.M. of x^3-3x^2+3x-1, x^3-x^2-x+1, and x^4-2x^3+2x-1.

237. Extract the square root of

$$\frac{x^2}{a^2}-\frac{2x}{a}+3-\frac{2a}{x}+\frac{a^2}{x^2}.$$

238. Solve the equations:

$$\frac{1}{ab-ax}+\frac{1}{bc-bx}=\frac{1}{ac-ax} \quad \ldots\ldots(1).$$

$$\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}+\frac{1}{z}&=6\\\frac{2}{x}-\frac{3}{y}+\frac{4}{z}&=8\\\frac{3}{x}-\frac{4}{y}+\frac{5}{z}&=10\end{aligned}\right\} \quad \ldots\ldots(2).$$

239. Two vessels contain mixtures of wine and water; in one there is twice as much wine as water and in the other three times as much water as wine. Find how much must be drawn off from each to fill a third vessel of 15 gallons, so that its contents may be half wine and half water.

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240. Multiply $x^{(m-1)n}-y^{(n-1)m}$ by x^n-y^m, and divide x^4+y^4 by $x^2-xy\sqrt{2}+y^2$.

241. If $x+y+z=0$, prove that $\frac{x^2}{yz}+\frac{y^2}{zx}+\frac{z^2}{xy}=3$.

242. Given $ax+by=m$, $bx-ay=n$, $a^2+b^2=1$; shew that

$$x^2+y^2=m^2+n^2.$$

243. Simplify $\frac{1}{(a-b)(c-a)}+\frac{1}{(b-c)(a-b)}+\frac{1}{(c-a)(b-c)}$.

244. Shew that

$$\left(\frac{1}{y-z}+\frac{1}{z-x}+\frac{1}{x-y}\right)^2=\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}+\frac{1}{(x-y)^2}.$$

245. Resolve into factors the expression

$$a(b-c)^2+b(c-a)^2+c(a-b)^2+8abc.$$

246. Find K when

$$(x-a)(x-3a)(x+a)(x+3a)+K$$

is a perfect square.

247. Extract the square root of

$$a^2x^2+6ac+\frac{12bc}{x}+4b(ax+b)+\frac{9c^2}{x^2};$$

and the cube root of

$$x^3-\frac{1}{x^3}-3x^2-\frac{3}{x^2}+5.$$

248. A boatman rows to a place 120 miles distant and back in 35 hours, and finds that he can row 4 miles with the current in the same time as 3 miles against the current; what is the rate of the current?

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249. By resolution into factors simplify

$$\frac{x^2-7xy+12y^2}{x^2+5xy+6y^2}+\frac{x^2-5xy+4y^2}{x^2+xy-2y^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\frac{(x^2-xy+y^2)^3+(x^2+xy+y^2)^3}{2(x^2+y^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

250. Find the H.C.F. and L.C.M. of

$$x^3-2x^2-19x+20,\ x^3+2x^2-23x-60,\ x^4+7x^3-4x^2-52x+48.$$

251. Solve the equation:

$$\frac{6}{7-\dfrac{6}{7-\dfrac{6}{7-x}}}=1.$$

252. What value of a will make the product of $3-8a$ and $3a+4$ equal to the product of $6a+11$ and $3-4a$?

253. The gross income of a man was £40 more in the second of two particular years than in the first, but in consequence of the income-tax rising from 4*d.* to 6*d.* in the pound in the second year his net income was unaltered; find his income in each year.

254. The sum of the ages of a man and his wife is six times the sum of the ages of their children; two years ago the sum of their ages was ten times the sum of the ages of the children, and six years hence the sum of their ages will be three times the sum of the ages of the children. How many children have they?

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255. Resolve into their simplest factors:

(1) $x^4-3a^2x^2+a^4$, (2) $x^3+z^3-3xz+1$, (3) $x^2+5xy-24y^2+x-3y$.

256. If $ax+by=1$ and $ax^2+by^2=\dfrac{1}{a+b}$, prove that

$$ax^n+by^n=\left(\frac{1}{a+b}\right)^{n-1}.$$

257. Find the square root of

$$a^6+\frac{1}{a^6}-6\left(a^4+\frac{1}{a^4}\right)+15\left(a^2+\frac{1}{a^2}\right)-20;$$

and the cube root of

$$1-6x+21x^2-44x^3+63x^4-54x^5+27x^6.$$

258. If p be the difference between any fraction and its reciprocal, q the difference between the square of the same fraction and the square of its reciprocal, shew that

$$p^2(p^2+4)=q^2.$$

259. The trinomial ax^2+bx+c becomes 8, 22, 42 respectively, when x has the value 2, 3, 4; what does it become when $x=-\frac{1}{3}$?

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260. Find the numerical value of

$$(xyz+x+y+z)^2-(1+yz+zx+xy)^2,$$

when $x=2$, $y=3$, $z=4$.

261. Find the H.C.F. of $2x^3-x^2-x-3$ and $x^5-x^3-4x^2-3x-2$; and the L.C.M. of $9x^2-4$, $4x^2-36$, $3x^2-7x-6$ and $3x^2+7x-6$.

262. Simplify

$$\sqrt{\left\{\frac{(a-b)^6+8ab(a-b)^4+16a^2b^2(a-b)^2}{a^6b^2-2a^2b^4+a^2b^6}\right\}} \quad \ldots\ldots\ldots(1).$$

$$x+1-\cfrac{x}{x+2-\cfrac{x+1}{x+\cfrac{1}{x+2}}} \quad \ldots\ldots\ldots(2).$$

263. Solve the equations:

$$\frac{x-2}{2}-\frac{x-5}{9}+\frac{5(x-1)}{6}=0 \quad \ldots\ldots\ldots(1).$$

$$\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7} \quad \ldots\ldots\ldots(2).$$

$$\left.\begin{aligned}\frac{3}{x}+\frac{6}{y}&=4\\ \frac{9}{x}-\frac{2}{y}&=2\end{aligned}\right\} \quad \ldots\ldots\ldots(3).$$

264. Find the fraction which becomes equal to $\frac{1}{2}$ when the numerator is increased by unity, and equal to $\frac{1}{3}$ when the denominator is increased by unity.

265. In a mile race between a bicycle and a tricycle their rates were proportional to 5 and 4; the tricycle had a minute's start but was beaten by 176 yards; find the rate of each.

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266. Prove without actual division that

$$5x^4+6x^3-7x^2-8x-480$$

is exactly divisible by $x-3$.

267. Divide

$$x^2(4n^3+16n^2+21n+9)+2xy(6n^3+23n^2+29n+12)$$
$$+y^2(8n^3+28n^2+32n+12)$$

by $(n+1)(2n+3)(x+2y)$.

268. Simplify $\dfrac{x^2}{ab}+\dfrac{(x-a)^2}{a(a-b)}-\dfrac{(x-b)^2}{b(a-b)}$.

269. Find the H.C.F. of

$$x^4+2x^3+3x^2+2x+1 \text{ and } x^5+5x^4+2x^3-4x-1.$$

270. Extract the cube root of

$$27x^6-108x^5+171x^4-136x^3+57x^2-12x+1.$$

271. Shew that

$$(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3-3(x^2-yz)(y^2-zx)(z^2-xy)$$

is an exact square.

272. Solve the equations:

$$\frac{x-4a}{x-3a}+\frac{x-5a}{x-4a}=\frac{x+6a}{x-4a}+\frac{x+5a}{x-3a} \quad \text{.....(1).}$$

$$\frac{x-ab}{c}+\frac{x-ac}{b}=\frac{2x}{b+c} \quad \text{.....(2).}$$

273. If of two numbers of two digits seven times the sum is eleven times the difference, and the numbers consist of the same digits reversed, find the numbers.

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274. Simplify

$$\left(1-\frac{1}{1+x}\right)\left(x+\frac{1}{2+x}\right)\frac{\frac{1}{x^2}-x}{1+\frac{1}{x}}\div\left(1+x+\frac{1}{x}\right) \quad \text{.....(1).}$$

$$\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x+1}-\frac{x}{x^2-1}+\frac{3}{x(x^2-1)} \quad \text{.....(2).}$$

275. If $x=\frac{1}{2}(a+b)$, $y=\frac{1}{2}(b+c)$ and $b^2=ac$, prove that

$$\frac{a}{x}+\frac{c}{y}=2.$$

276. Find the H.C.F. of $x^3+9x^2+26x+24$ and $x^3+9x^2+23x+15$; and the L.C.M. of x^2-xy^2, x^2y+xy^2 and $xy-y^2$.

277. Extract the square root of

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{x}{y}+\frac{y}{x}-1\tfrac{3}{4};$$

and the cube root of

$$x^3-\frac{27}{x^3}-6\left(x^2+\frac{9}{x^2}\right)+3\left(x-\frac{3}{x}\right)+28.$$

278. Solve the equations:

$$\frac{1-x^2}{1-4x^2}-\frac{x}{2x-1}+\frac{1}{4}=0 \quad \text{.....(1).}$$

$$\left.\begin{aligned}\frac{2}{x}-\frac{3}{2y}&=\frac{41}{35}\\ \frac{2\frac{1}{2}}{2x}+\frac{3\frac{1}{2}}{y}&=-\frac{73}{70}\end{aligned}\right\} \quad \text{.....(2).}$$

279. A labourer is engaged for 30 days on condition that he receives half-a-crown for each day he works and forfeits a shilling for each day he is idle, and he receives £2. 7*s.* altogether; how many days did he work?

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280. Find the factors of

(1) $a^4+2a^2b^2+4b^4$, (2) $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

281. Shew that the sum of the coefficients of x^3 and x^9 in the cube of

$$1-x+x^3+x^4$$

is zero.

282. Determine the values of c which make $3x^2+x-4$ and $12x^2-5x+c$ have a common measure.

283. Simplify $\dfrac{2x^3+29x^2+62x+24}{2x^3+25x^2+11x-12}$.

284. If $x+y+z-1=\sqrt{2\,(1-x)\,(1-y)\,(1-z)}$,
shew that $x^2+y^2+z^2+2xyz-1=0$.

285. Prove that

$$\frac{(b^2-c^2)^3+(c^2-a^2)^3+(a^2-b^2)^3}{(b-c)^3+(c-a)^3+(a-b)^3}=(b+c)\,(c+a)\,(a+b).$$

286. Shew that

$$x^{2n}+a^{2n}+ax\,(x^{2n-2}+a^{2n-2})-a^{n-1}x^{n-1}\,(x+a)^2$$

is divisible by x^2-a^2 when n is an even integer.

287. Extract the square root of

$$x^6-8x^5+20x^4-22x^3+28x^2-12x+9.$$

288. Find the cube root of

$$8x^6+12x^5+18x^4+13x^3+9x^2+3x+1;$$

and hence solve the equation

$$8x^6+12x^5+18x^4+13x^3+9x^2+3x=7.$$

289. If $x+y+z=0$, prove that

$$x\,(y-z)^2+y\,(z-x)^2+z\,(x-y)^2+9xyz=0.$$

290. Find a number of two digits such that the sum of the digits is a multiple of 9, and if 27 be added to the number the order of the digits is reversed.

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291. Find the value of $\sqrt{(x^2-2xy+4y^2)}$ in terms of a and b, where $x=9a^2-12ab$ and $y=2b^2-6ab$.

292. Find the H.C.F. of

$$5x^2-19xy+12y^2 \text{ and } 3x^3-10x^2y+4xy^2-3y^3;$$

and the L.C.M. of $2x^4+x^3+3x^2+x+1$ and $2x^4+x^3-x^2-x-1$.

293. Prove that

$$(bz-cy)^2+(cx-az)^2+(ay-bx)^2+(ax+by+cz)^2$$
$$=(a^2+b^2+c^2)\,(x^2+y^2+z^2).$$

294. The value of the trinomial ax^2+bx-c becomes 46 when $x=4$, 26 when $x=3$, and 12 when $x=2$; what are the values of a, b, c?

295. Shew that the value of the expression

$$y^2z-z^2y+z^2x-x^2z+x^2y-y^2x$$

is not altered if any the same quantity be added to or subtracted from each of the quantities x, y, z.

296. If $2s=a+b+c+d$, prove that

$$4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2=16(s-a)(s-b)(s-c)(s-d).$$

297. If $x=a^2-bc$, $y=b^2-ca$, $z=c^2-ab$, shew that

$$ax+by+cz=(a+b+c)(x+y+z).$$

298. Find a value of x which will make the expression

$$x^5-8x^3+11x^2+7x-1789$$

exactly divisible by x^2+7x-1.

299. Prove that the product of any four consecutive even integers increased by 16 is a perfect square.

300. Given that $x=\frac{b+c-a}{2}$, $y=\frac{c+a-b}{2}$, $z=\frac{a+b-c}{2}$, shew that

$$x^3+y^3+z^3-3xyz=\tfrac{1}{2}(a^3+b^3+c^3-3abc).$$

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301. Subtract $bcd^2-(a^2-b^2)bd$ from $(a^2+bc)d^2-(a^2-b^2)bd$, and multiply $1-x+x^2-x^3$ by $1+x+x^2+x^3$.

302. Solve the equations:

$$10\left(x+\frac{1}{2}\right)-23=6x\left(\frac{1}{x}-\frac{1}{3}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$5+7\sqrt{(\tfrac{2}{3}x-6)}=19 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

303. I bought 25 yards of cloth for Rs. 223. 8as., for a part I paid Rs. 8. 8as. a yard and for the rest Rs. 9. 8as. a yard; how many yards were there at each price?

304. In a right-angled triangle the base is 8 and the sum of the hypotenuse and perpendicular is 12; find them.

305. A person has two horses and a saddle worth Rs. 75; if the saddle be put on the first horse, his value becomes double that of the second; but if the saddle be put on the second horse, his value will not amount to that of the first horse by Rs. 350; what is the value of each horse?

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306. Shew that

$$\{(ax+by)^2+(ay-bx)^2\}\times\{(ax+by)^2-(ay+bx)^2\}=(a^4-b^4)(x^4-y^4).$$

307. Divide $x^6+2x^3y^3+y^6$ by $(x+y)^2$.

308. Resolve $x^{12}-a^{12}$ into its simplest factors.

309. Simplify $\dfrac{1+\dfrac{a-b}{a+b}}{1-\dfrac{a-b}{a+b}} \div \dfrac{1+\dfrac{a^2-b^2}{a^2+b^2}}{1-\dfrac{a^2-b^2}{a^2+b^2}}$.

310. Find the H.C.F. of x^3+3x^2-9x+5 and $x^3-19x+30$.

311. Solve the equations:

$$\frac{1}{2}\left(x-\frac{a}{3}\right)-\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=0 \quad \text{.....(1).}$$

$$\frac{a}{x-a}-\frac{b}{x-b}=\frac{a-b}{x-c} \quad \text{.....(2).}$$

312. There are three numbers such that the sum of the first and second divided by their product is $\frac{1}{2}$, the sum of the second and third divided by their product is $\frac{1}{3}$, and the sum of the third and first divided by their product is $\frac{1}{4}$; find the numbers.

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313. Simplify the following:

$$\frac{1}{x+3}+\frac{x+1}{x^2-3x+9}-\frac{2x^2+x+12}{x^3+27} \quad \text{.....(1).}$$

$$\frac{a+c}{(b-a)(x-a)}+\frac{b+c}{(a-b)(x-b)} \quad \text{.....(2).}$$

$$\frac{a^4-b^4}{a^2-2ab+b^2}\times\frac{a-b}{a(a+b)} \quad \text{.....(3).}$$

314. Solve the equations:

$$\frac{1}{x-1}+\frac{1}{x-4}=\frac{2}{x-3} \quad \text{.....(1).}$$

$$\frac{2x+11}{x+5}-\frac{9x-9}{3x-4}=\frac{4x+13}{x+3}-\frac{15x-47}{3x-10} \quad \text{.....(2).}$$

315. A man bought a picture at a certain price and paid the same price for the frame; if the frame had cost £1 less and the picture 15*s.* more, the price of the frame would have been only half that of the picture; find the cost of the picture.

316. Find a fraction such that if 1 be subtracted from its numerator the value shall be $\frac{3}{4}$, and if 6 be added to the denominator the value shall be $\frac{1}{2}$.

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317. Divide $28x^4+13x^2y^2-xy^3+15y^4$ by $4x^2+4xy+3y^2$.

318. Reduce to its simplest form

$$\frac{1}{4a^3(a+x)}+\frac{1}{4a^3(a-x)}+\frac{1}{2a^2(a^2+x^2)}.$$

319. Solve the equations:

$$6\tfrac{1}{3} - \frac{x-7}{3} = \frac{4x-2}{5} \quad \ldots\ldots(1).$$

$$\sqrt{x+9} = 1+\sqrt{x} \quad \ldots\ldots(2).$$

$$\left.\begin{aligned} x+3y &= 10 \\ 3x+2y &= 9 \end{aligned}\right\} \quad \ldots\ldots(3).$$

320. A and B can do a piece of work in 30 days, C and A in 40 days and B and C in 50 days, and all three work together for 10 days; if two be then taken away, how long will each of the others take to finish it?

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321. Reduce to its simplest form

$$\frac{x+y}{x-y} + \frac{x-y}{x+y} - \frac{x^2+y^2}{x^2-y^2}.$$

322. Divide unity by $(a+b)^2$, giving three terms of the quotient and the remainder.

323. Solve the equations:

$$2x+11 = 7x-14 \quad \ldots\ldots(1).$$

$$\sqrt{x^2+9} = x+1 \quad \ldots\ldots(2).$$

$$\frac{a-b}{x-c} = \frac{a+b}{x+2c} \quad \ldots\ldots(3).$$

324. What fraction is that which if 1 be added to the numerator becomes 1, and if 1 be added to the denominator becomes $\frac{1}{2}$?

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325. Prove that

$$\frac{x+y}{y} - \frac{x}{x+y} - \frac{x^3-x^2y}{x^2y-y^3} = 1.$$

326. Divide a^8-x^8 by $a+x$.

327. Solve the equations:

$$\frac{2x}{5} + \frac{x-2}{3} = 2x-7 \quad \ldots\ldots(1).$$

$$4-\sqrt{3x} = \sqrt{4+3x} \quad \ldots\ldots(2).$$

$$\left.\begin{aligned} 2x - \frac{y-3}{5} &= 4 \\ 3y + \frac{x-2}{3} &= 9 \end{aligned}\right\} \quad \ldots\ldots(3).$$

328. A post is a fourth of its length in the mud, a third of its length in the water and ten feet above the water; what is its length?

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329. Add together

$x^2-(x-y+z)(x+y-z)$, $y^2-(y-z+x)(y+z-x)$, $z^2-(z-x+y)(z+x-y)$.

330. Multiply $x+y+z-\sqrt{yz}-\sqrt{zx}-\sqrt{xy}$ by $\sqrt{x}+\sqrt{y}+\sqrt{z}$, and divide $x^8+a^4x^4+a^8$ by x^2-ax+a^2.

331. Simplify the expression

$$\frac{1}{2(x-1)}-\frac{x-5}{x^2-7x+10}+\frac{x-6}{2(x^2-9x+18)}.$$

332. Solve the equations:

$$\frac{x-1}{3}-\frac{x-9}{2}+\frac{3x-2(x-2)}{7}=4\tfrac{1}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\left.\begin{aligned}5x+11y&=146\\11x+5y&=110\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

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333. Divide the continued product of $1+x+y$, $1+x-y$, $1-x+y$ and $x+y-1$ by $1+2xy-x^2-y^2$.

334. Resolve $4(uz-xy)^2-(u^2-x^2-y^2+z^2)^2$ into four factors.

335. Find the H.C.F. of $2x^5-11x^2-9$ and $4x^5+11x^4+81$; and reduce to its lowest terms the fraction $\dfrac{x^3-6x^2-37x+210}{x^3+4x^2-47x-210}$.

336. Simplify the following:

$$\frac{x^3}{(x-y)(x-z)}+\frac{y^3}{(y-z)(y-x)}+\frac{z^3}{(z-x)(z-y)} \quad \ldots\ldots\ldots(1).$$

$$\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-z)(y-x)}+\frac{1}{z(z-x)(z-y)} \quad \ldots\ldots\ldots(2).$$

$$\frac{x^2-yz}{(x-y)(x-z)}+\frac{y^2+zx}{(y+z)(y-x)}+\frac{z^2+xy}{(z-x)(z+y)} \quad \ldots\ldots\ldots(3).$$

337. If $x=\dfrac{4ab}{a+b}$, find the value of

$$\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}.$$

338. Solve the equations:

$$\frac{x-a}{b}+\frac{x-b}{c}+\frac{x-c}{a}=\frac{x-(a+b+c)}{abc} \quad \ldots\ldots\ldots\ldots(1).$$

$$\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7} \quad \ldots\ldots\ldots\ldots\ldots(2).$$

$$\sqrt{\left(\frac{a^2}{x}+b\right)}-\sqrt{\left(\frac{a^2}{x}-b\right)}=\sqrt{c} \quad \ldots\ldots\ldots\ldots\ldots(3).$$

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339. If $a=1$, $b=2$, $c=-\frac{1}{2}$, $d=0$, find the value of

$$\frac{a-b+c}{a-b-c}-\frac{ad-bc}{bd-ac}-\sqrt{\left(\frac{b^3}{a^3}-\frac{a^3}{c^3}\right)}.$$

340. Divide $6a^3-17a^2b+7ab^2-5b^3$ by $2a-5b$, and find the continued product of $ax-1$, $ax+1$, $x^2+\frac{1}{a^2}$.

341. Find the H.C.F. of $x^3+6x^2+11x+6$ and $x^3+9x^2+27x+27$, and the L.C.M. of xy, $x-y$ and y^3-x^2y.

342. Simplify the following:

$$\frac{x+3y}{4(x+y)(x+2y)}+\frac{x+2y}{(x+3y)(x+y)}-\frac{x+y}{4(x+2y)(x+3y)} \quad \ldots (1).$$

$$\frac{a^2+3a+2}{a^2+2a+1}\times\frac{a^2+5a+4}{a^2+7a+10} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

343. Extract the cube root of

$$x^6+6x^5+21x^4+44x^3+63x^2+54x+27.$$

344. Solve the equations:

$$\frac{x-3}{7}-\frac{\frac{x}{2}-3}{3}=\frac{\frac{x}{6}+2}{2}-\frac{x-6}{3}+\frac{x}{8} \quad \ldots\ldots\ldots\ldots (1).$$

$$\frac{x-a}{b-a}=\frac{x-b}{a-b} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$\frac{p-q}{qx+r}=\frac{p+q}{px-r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

345. A is twice as old as B and 4 years older than C, and the sum of the ages of A, B and C is 96 years; find B's age.

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346. Find the product of the four factors

$$x+y+z,\quad y+z-x,\quad z+x-y,\quad x+y-z.$$

347. Divide $(x+y+z)(yz+zx+xy)-xyz$ by $x+y$.

348. Reduce to its simplest form

$$\frac{x^2-(y-z)^2}{(z+x)^2-y^2}+\frac{y^2-(z-x)^2}{(x+y)^2-z^2}+\frac{z^2-(x-y)^2}{(y+z)^2-x^2}.$$

349. Extract the square root of

$$\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12.$$

350. Shew that

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}=1.$$

351. Solve the equations:

$$\frac{5-3x}{4}+\frac{5x}{3}=\frac{3}{2}-\frac{3-5x}{3} \quad \text{.....(1).}$$

$$\cdot 5x+\frac{\cdot 02x+\cdot 07}{\cdot 03}-\frac{x+2}{9}=9\cdot 5 \quad \text{.....(2).}$$

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352. Find the H.C.F. of $2x^3+9x^2+4x-15$ and $4x^3+8x^2+3x+20$.

353. Reduce to its lowest terms $\dfrac{x^4-x^3-x+1}{x^4+x^3-x-1}$.

354. Simplify $\left(\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}\right)\div\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)$.

355. Prove the identities:

$$\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right);$$

$$(a+b+c)^3abc-(bc+ca+ab)^3=(a^2-bc)(b^2-ca)(c^2-ab).$$

356. If $2s=a+b+c$, shew that

$$16s(s-a)(s-b)(s-c)=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4.$$

357. Solve the equations:

$$(x+\tfrac{5}{2})(x-\tfrac{3}{2})-(x+5)(x-3)+\tfrac{3}{4}=0 \quad \text{.....(1).}$$

$$\frac{\sqrt{x}+a}{(\sqrt{x}-b)(\sqrt{x}-c)}+\frac{\sqrt{x}+b}{(\sqrt{x}-c)(\sqrt{x}-a)}+\frac{\sqrt{x}+c}{(\sqrt{x}-a)(\sqrt{x}-b)}=0 \quad \text{(2).}$$

358. Solve the simultaneous equations:

$$\left.\begin{aligned} ax+by+c&=0\\ a_1x+b_1y+c_1&=0\end{aligned}\right\} \quad \text{.....(1).}$$

$$\left.\begin{aligned} x+5y-4z&=5\\ 3x-2y+2z&=14\\ -10x+8y+z&=6\end{aligned}\right\} \quad \text{.....(2).}$$

359. Extract the square root of

$$x^6+8x^4-2x^3+16x^2-8x+1.$$

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360. Given $a=\sqrt{2}$, $b=\sqrt{3}$, $c=\sqrt{4}$ and $d=0$; find the value of

$$\sqrt{\{(a^2+b^2+c^2)(b^2+c^2)(b^2+d^2)\}}.$$

361. Find the H.C.F. of x^3+4x^2-5 and x^3-3x+2, and the L.C.M. of $x^5-5x^3+x^2+4x-4$ and $x^4+x^3-6x^2-4x+8$.

362. Simplify $\left(\dfrac{a+b}{a-b}+\dfrac{a^2+b^2}{a^2-b^2}\right)\div\left(\dfrac{a-b}{a+b}-\dfrac{a^3-b^3}{a^3+b^3}\right)$.

363. Shew that

$$1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2=\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}.$$

364. Extract the square root of

$$a^2+b^2+c^2+d^2-2a(b-c+d)-2b(c-d)-2cd.$$

365. Solve the equations:

$$\frac{4x+3}{9}+\frac{29-7x}{12-5x}=\frac{8x+19}{18} \quad \text{.....(1).}$$

$$\frac{8x+4}{\sqrt{x+5}}=4\sqrt{x+5} \quad \text{.....(2).}$$

$$\left.\begin{aligned}\tfrac{1}{2}x+\tfrac{1}{3}y+\tfrac{1}{6}z&=12\\ \tfrac{1}{2}y+\tfrac{1}{3}z-\tfrac{1}{6}x&=8\\ \tfrac{1}{3}z+\tfrac{1}{2}x&=10\end{aligned}\right\} \quad \text{.....(3).}$$

366. There is a number of two digits the sum of whose digits is 5, and if 10 times the digit in the place of tens be added to 4 times the digit in the place of units the number will be reversed; what is the number?

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367. Divide $x^3+y^3+3xy-1$ by $x+y-1$.

368. By resolution into factors find the H.C.F. of x^4+2x^2+1, $x^6+x^4-x^2-1$ and x^4-1; and the L.C.M. of $6x^2-x-1$, $3x^2+7x+2$ and $2x^2+3x-2$.

369. Simplify:

$$\frac{x}{x-a}-\frac{x}{x+a}-\frac{\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}}{\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}} \quad \text{.....(1).}$$

$$\frac{a^2+ac}{a^2c-c^3}-\frac{a-c}{(a+c)c}-\frac{2c}{a^2-c^2} \quad \text{.....(2).}$$

$$\frac{3x^3-2x^2-x}{4x^3-2x^2-3x+1} \quad \text{.....(3).}$$

370. Extract the square root of

$$x^4-3x^3+\tfrac{11}{12}x^2+2x+\tfrac{4}{9}.$$

371. Solve the equations:

$$\tfrac{1}{8}(x-2)-\tfrac{1}{7}(x-4)=\tfrac{1}{12}(2x-3)-2\tfrac{3}{4} \quad \text{.....(1).}$$

$$\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}&=m\\ \frac{b}{x}+\frac{a}{y}&=n\end{aligned}\right\} \quad \text{.....(2).}$$

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372. Find the product of $3a+2b$ and $3a-3b+2c$, and test the result by making $a=1$, $b=c=3$.

373. Divide $x^8+x^6y^2+x^4y^4+x^2y^6+y^8$ by $x^4-x^3y+x^2y^2-xy^3+y^4$, and $\frac{a^3}{b^3}+\frac{b^3}{a^3}$ by $\frac{a}{b}+\frac{b}{a}$.

374. Prove that

$$\frac{1}{1+\frac{1}{a\div x}}+\frac{1}{1-\frac{1}{a\div x}}+\frac{2}{1+\frac{1}{a^2\div x^2}}=\frac{4a^4}{a^4-x^4}.$$

375. Simplify (1) $\dfrac{x^{a+b}\,x^{a-b}\,x^{c-2a}}{x^{c-a}}$,

and (2) $$\frac{\frac{1+x}{1-x}+\frac{4x}{1+x^2}+\frac{8x}{1-x^2}-\frac{1-x}{1+x}}{\frac{1+x^2}{1-x^2}+\frac{4x^2}{1+x^4}-\frac{1-x^2}{1+x^2}}.$$

376. Find the L.C.M. of $x^3+x^2y+xy^2+y^3$ and $x^3-x^2y+xy^2-y^3$.

377. Reduce to its lowest terms

$$\frac{2x^4-x^3-9x^2+13x-5}{7x^3-19x^2+17x-5}.$$

378. Extract the square root of

$$4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4.$$

379. Solve the equations:

$$\frac{x-3}{5}-\frac{x-5}{4}=\frac{2}{3} \quad \text{.....(1).}$$

$$\frac{a-x}{a}+\frac{2a-x}{2a}=\frac{3a-x}{3a} \quad \text{.....(2).}$$

$$\left.\begin{aligned}\frac{1}{5x}+\frac{y}{9}&=5\\ \frac{1}{3x}+\frac{y}{2}&=14\end{aligned}\right\} \quad \text{.....(3).}$$

380. AB is a railway 220 miles long, and three trains P, Q, R travel upon it at the rate of 25, 20, 30 miles per hour respectively; P and Q leave A at 7 A.M. and 8.15 A.M. respectively, and R leaves B at 10.30 A.M.; when and where will P be equidistant from Q and R?

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381. Multiply $x^3-\frac{1}{2}x^2y-3y^3$ by $2x^2-\frac{1}{3}y^2$.

382. Find the L.C.M. of $2(x-2)^2$, $2x^2-8$, x^2+2x and $2x^2-4x$.

383. Reduce $\dfrac{10x^3+19x^2-9}{25x^3-19x+6}$ to its lowest terms.

384. Simplify :

$$1+\frac{a}{b}-\frac{b}{a+b}-\frac{a^2}{ab-b^2}+\frac{2a^2}{a^2-b^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\left(1-\frac{1}{1+x}\right)\left(x+\frac{1}{2+x}\right)\times\frac{\frac{1}{x^2}-x}{1+\frac{1}{x}}\div\left(1+x+\frac{1}{x}\right) \quad \ldots\ldots\ldots\ldots(2).$$

385. Extract the square root of

$$x^4-2x^2+3-\frac{2}{x^2}+\frac{1}{x^4}.$$

386. Solve the equations :

$$\frac{6}{3x-5}-\frac{1}{x-5}=\frac{2}{2x-5} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\left.\begin{aligned} 4x-\tfrac{3}{5}(2y-3)&=6\tfrac{1}{5} \\ 3y-\tfrac{2}{3}(3x-1)&=7 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

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387. Divide x^4-10x^2+9 by x^2-2x-3.

388. Find the H.C.F. of

$3x^3-17x^2+19x+11$ and $6x^3-25x^2+17x-22$.

389. Simplify

$$\left(\frac{2a}{x^2-a^2}-\frac{1}{x-a}+\frac{2}{x+a}\right)\times\frac{x}{x-a+\frac{a^2}{x}}.$$

390. Extract the square root of

$$a^4+b^4+c^4+d^4-2(a^2+c^2)(b^2+d^2)+2a^2c^2+2b^2d^2.$$

391. Solve the equations :

$$\frac{7x-1}{4}-\frac{1}{3}\left\{2x-\frac{1-x}{2}\right\}=6\tfrac{1}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\left.\begin{aligned} \frac{x-y}{3}&=\frac{y-1}{4} \\ \frac{4x-5y}{7}&=x-7 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

392. Rs. 1100 are so divided among A, B and C, that if A were to give B Rs. 200, B would then have twice as much as A and three times as much as C; how many rupees did A, B and C each receive originally ?

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393. Reduce to their simplest forms:

(1) $\frac{3x}{2}\sqrt{\frac{400y^2}{81x^2}}$; (2) $\frac{(x^3-y^3)(x+y)^2}{(x^2+xy+y^2)(x^2-y^2)}$;

(3) $\frac{2+x}{2(1+x)}-\frac{2-x}{2(1-x)}+\frac{x}{1+x^2}$.

394. Find the H.C.F. of $x^4-9a^2x^2+10a^3x$ and $ax^3-a^2x^2-4a^4$; and the L.C.M. of $3ax^2-3a^2x$, x^2-a^2, x^2+ax, $\sqrt{3ax}$, $\sqrt{x}-\sqrt{a}$.

395. Extract the square root of

$$x^4-x^3+\frac{x^2}{4}+4x-2+\frac{4}{x^2}.$$

396. Solve the equations:

$$\frac{12}{x+2}=6-2\frac{3x+2}{x+1} \quad \text{.....(1).}$$

$$\sqrt{x}+\sqrt{4+x}=\frac{2}{\sqrt{x}} \quad \text{.....(2).}$$

$$\left.\begin{aligned} 2x-\frac{2y-1}{3}&=3\tfrac{5}{24}+\frac{3x-2y}{4}\\ 4y-\frac{5-2x}{4}&=6-\frac{3-2y}{5}\end{aligned}\right\} \quad \text{.....(3).}$$

397. Two persons started at the same time from A; one rode on horseback at the rate of $7\frac{1}{2}$ miles an hour and arrived at B 30 minutes later than the other, who travelled the same distance by train at the rate of 30 miles an hour; find the distance between A and B.

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398. Simplify:

(1) $\frac{x^{m+2n}\,x^{3m-8n}}{x^{5m-6n}}$; (2) $\frac{a}{a+b}-\frac{a+b}{2b}+\frac{a^2+b^2}{2ab-2b^2}$;

(3) $\frac{\frac{a^3}{b^3}-\frac{b^3}{a^3}}{\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{a}{b}+\frac{b}{a}-1\right)}\times\frac{\frac{1}{b}-\frac{1}{a}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{ab}}$.

399. Find the L.C.M. of $1+4x+4x^2-16x^4$ and $1+2x-8x^3-16x^4$.

400. Extract the square root of

$$9x^4-2x^3y+1\tfrac{6}{9}3x^2y^2-2xy^3+9y^4.$$

401. The expression $ax-3b$ is equal to 30 when x is 3 and to 24 when x is 7; what is its value when x is 4·3, and for what value of x is it zero?

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402. Subtract $3a-\frac{2}{3}b+\frac{3}{4}c$ from $2a+\frac{1}{3}b-\frac{1}{4}c$; and multiply

$$\frac{x^2}{y^2}+1+\frac{y^2}{x^2} \text{ by } \frac{x}{y}-\frac{y}{x}.$$

403. Find the H.C.F. of

$$6x^4+x^3-6x^2-5x-2 \text{ and } 2x^4+3x^3+2x^2-7x-6.$$

404. Simplify

$$\frac{1+x+x^2}{1-x^3}+\frac{1-x+x^2}{1+x^3}-\left(\frac{x}{1+x}+\frac{1-x}{x}\div\frac{1+x}{x}\right)\times\frac{1}{1-x}.$$

405. If $\frac{b-c}{a}+\frac{c+a}{b}+\frac{a-b}{c}=1$ and $a-b+c$ is not $=0$, shew that

$$\frac{1}{a}=\frac{1}{b}+\frac{1}{c}.$$

406. Solve the equations:

$$\frac{15-\frac{2}{3}x}{5}-\frac{2x+5}{2\frac{1}{2}}=\frac{17-\frac{14}{3}x}{3} \quad \ldots\ldots(1).$$

$$\frac{x-\frac{1}{2}}{5}-\frac{7x-3}{6}+\frac{3}{7}=0 \quad \ldots\ldots(2).$$

$$\sqrt{(5x-1)}=1+\sqrt{(5x-2)} \quad \ldots\ldots(3).$$

$$\left.\begin{aligned} 4x-\tfrac{1}{3}(5y-4)&=1\\ \tfrac{1}{3}x+\tfrac{1}{4}(3y-2x)&=\tfrac{1}{2}\end{aligned}\right\} \quad \ldots\ldots(4).$$

407. How many bundles of hay at Rs. 5 per thousand must a hay-merchant mix with 5600 bundles at Rs. 6 per thousand in order that he may gain 20 per cent. by selling the whole at 11 annas per hundred?

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408. Simplify the following expressions:

$$3a-[a+b-2\{a+b+c-(a-b+c-d)\}+a] \quad \ldots\ldots(1).$$

$$(x-y)^3+(x+y)^3+3(x-y)^2(x+y)+3(x+y)^2(x-y) \quad \ldots\ldots(2).$$

$$\frac{\frac{a}{a-b}-\frac{a}{a+b}}{\frac{b}{a-b}-\frac{b}{a+b}}\div\frac{\frac{a+b}{a-b}+\frac{a-b}{a+b}}{\frac{a+b}{a-b}-\frac{a-b}{a+b}}\times\frac{a^2}{a^2+b^2} \quad \ldots\ldots(3).$$

409. Find the H.C.F. of $2x^3+3x^2+2x-2$ and $4x^4-2x^3+2x-1$.

410. Multiply x^2-x+1 by $\frac{1}{x^2}+\frac{1}{x}+1$.

411. Extract the square root of

$$4-4c+2b+c^2-bc+\tfrac{1}{4}b^2.$$

412. Solve the equations:

$$\frac{2x-13}{9}-\frac{x-1}{11}=\frac{x}{8}+\frac{x}{7}-9 \quad \text{.....(1).}$$

$$\left.\begin{aligned}\frac{x+y}{2}+\frac{3x-5y}{4}&=2\\ \frac{x}{14}+\frac{y}{18}&=1\end{aligned}\right\} \quad \text{.....(2).}$$

413. A can do a piece of work in 9 days, B in twice that time, C can only do three-quarters as much as A in a day; how long would A, B and C, working together, require to do the piece of work?

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414. Divide $x^4+x^3-24x^2-35x+57$ by x^2+2x-3.

415. Simplify $\dfrac{x+2}{1+x+x^2}-\dfrac{x-2}{1-x+x^2}-\dfrac{2x^2-4}{1-x^2+x^4}$.

416. Find the H.C.F. of $x^4+x^3-11x^2-9x+18$ and

$$x^4-10x^3+35x^2-50x+24.$$

417. Find the first four terms of the square root of a^2+x^2, and from the result deduce the square root of 101 correct to six places of decimals.

418. Solve the equations:

$$\frac{2x-3}{6}+\frac{3x-8}{11}=\frac{4x+15}{33}+\frac{1}{2} \quad \text{.....(1).}$$

$$2(x+2)=1+\sqrt{(4x^2+9x+14)} \quad \text{.....(2).}$$

$$\left.\begin{aligned}3x+4y-11&=0\\ 5y-6z+8&=0\\ 7z-8x-13&=0\end{aligned}\right\} \quad \text{.....(3).}$$

419. A and B together can do a piece of work in 15 days, and A can do it alone in 24 days; how long would B take to do it alone?

420. Two passengers have together 5 cwt. of luggage and are charged for the excess above the weight allowed 5*s.* 2*d.* and 9*s.* 10*d.* respectively; but if the luggage had all belonged to one of them he would have been charged 19*s.* 2*d.* How much luggage is each passenger allowed to carry free of charge? And how much had each passenger?

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421. Divide

$$x(1+y^2)(1+z^2)+y(1+z^2)(1+x^2)+z(1+x^2)(1+y^2)+4xyz$$

by

$$1+yz+zx+xy.$$

422. If $\dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c}$, evaluate

$$(b-c)x+(c-a)y+(a-b)z.$$

423. Simplify the expression

$$\frac{1}{(4x^3-3x)^2}-\left(\frac{\frac{3\sqrt{1-x^2}}{x}-\frac{\sqrt{(1-x^2)^3}}{x^3}}{1-3\frac{1-x^2}{x^2}}\right)^2.$$

424. Extract the square root of

$$(a^2+b^2+c^2)(x^2+y^2+z^2)-(bz-cy)^2-(cx-az)^2-(ay-bx)^2.$$

425. Solve the equations:

$$\frac{4x+3}{9}+\frac{13x}{108}=\frac{8x+19}{18} \quad \text{.....(1).}$$

$$\sqrt{\{4x^2+20x+17+\sqrt{(16x^2+11x+10)}\}}=-2(x+2) \quad \text{.....(2).}$$

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426. Multiply $a^{2n}-a^nx^n+x^{2n}$ by a^n+x^n,

and divide $x^{2^n}-y^{2^n}$ by $x^{2^{n-1}}+y^{2^{n-1}}$.

427. Find the H.C.F. of $x^2+\frac{7}{6}x+\frac{1}{3}$ and $x^2+\frac{2}{3}x+\frac{1}{12}$.

428. Simplify $\dfrac{x-y}{x-z}+\dfrac{x-z}{x-y}-\dfrac{(y-z)^2}{(x-y)(x-z)}$.

429. Solve the equations:

$$x-k+\sqrt{k^2+x^2}=m \quad \text{.....(1).}$$

$$\left.\begin{aligned} a^x\times a^{y+1}&=a^7\\ a^{2y}\times a^{3x+5}&=a^{20}\end{aligned}\right\} \quad \text{.....(2).}$$

$$\left.\begin{aligned} \frac{4}{x}+\frac{10}{y}&=2\\ \frac{3}{x}+\frac{2}{y}&=\frac{19}{20}\end{aligned}\right\} \quad \text{.....(3).}$$

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430. Simplify $\left\{\dfrac{x}{a}+\dfrac{2x^2}{a(b-x)}\right\}\left\{\dfrac{a}{x}-\dfrac{2ax}{x(b+x)}\right\}$.

431. Find the H.C.F. and L.C.M. of

$$3x^2-10ax+7a^2 \text{ and } x^3-5ax^2+7a^2x-3a^3.$$

432. Extract the square root of

$$x^8-2\frac{x^{11}}{a^3}+2a^4x^4+\frac{x^{14}}{a^6}-2ax^7+a^8.$$

433. Solve the equations:

$$4x-\frac{x-1}{2}=x+\frac{2x-2}{5}+24 \quad \text{.....(1).}$$

$$15+\sqrt{(x+7)}=19 \quad \text{.....(2).}$$

$$\left.\begin{aligned}\frac{7+x}{5}-\frac{2x-y}{4}&=3y-5\\ \frac{5y-7}{2}+\frac{4x-3}{6}&=18-5x\end{aligned}\right\} \quad \text{.....(3).}$$

434. A boat goes up-stream 30 miles and down-stream 44 miles in 10 hours, and it also goes up-stream 40 miles and down-stream 55 miles in 13 hours; find the rate of the stream and of the boat.

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435. Simplify $\dfrac{1}{abx}+\dfrac{1}{a(a-b)(x-a)}+\dfrac{1}{b(b-a)(x-b)}$.

436. Resolve into their elementary factors:

(1) $x^2-5ax-66a^2$, (2) $(1-c^2)(1+a)^2-(1-a^2)(1+c)^2$.

437. A man receives $\frac{x}{y}$ths of Rs. 10 and afterwards $\frac{y}{x}$ths of Rs. 10, and he then gives away Rs. 20; shew that he cannot lose by the transaction.

438. Solve the equations:

$$\sqrt{x^2+11x+20}-\sqrt{x^2+5x-1}=3 \quad \text{.....(1).}$$

$$\frac{4{\cdot}05}{9x}-\frac{{\cdot}3}{{\cdot}8-2x}=\frac{1{\cdot}8}{x}-\frac{3{\cdot}6}{2{\cdot}4-6x} \quad \text{.....(2).}$$

$$\left.\begin{aligned}ax+by&=c\\ a^2x+b^2y&=c^2\end{aligned}\right\} \quad \text{.....(3).}$$

439. *A* challenged *B* to ride a bicycle race of 1040 yards; he first gave *B* 120 yards start but lost by 5 seconds, and he then gave *B* 5 seconds start and won by 120 feet. How long does each take to ride the distance?

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440. Find the continued product of $x-a$, x^2+ax+a^2 and x^3+a^3.

441. Resolve into factors:

(1) $x^2+13x+42$; (2) x^2+x-42; (3) $343x^3+512y^3$.

442. Find the H.C.F. of $x^3-7x^2-80x+576$ and $3x^2-14x-80$, and the L.C.M. of these two expressions and $3x^2+17x-90$.

443. Solve the equations:

$$(6x+9)^2+(8x-7)^2=(10x+3)^2-71 \quad \ldots\ldots(1).$$

$$\cdot 65x+\frac{\cdot 585x-\cdot 975}{\cdot 6}=\frac{1\cdot 56}{\cdot 2}-\frac{\cdot 39x-\cdot 78}{\cdot 9} \quad \ldots\ldots(2).$$

$$\left.\begin{aligned}\frac{x-2}{2}-\frac{x+y}{14}&=\frac{x-y-1}{8}-\frac{y+12}{4}\\ \frac{x+7}{3}+\frac{y-5}{10}&=1-x-\frac{5(y+1)}{7}\end{aligned}\right\} \quad \ldots\ldots(3).$$

444. The distance from a place P to another place Q is $3\frac{1}{2}$ miles; two persons A and B start together from P to go to Q, the former by carriage which travels at the rate of 6 miles an hour, and the latter walking at the rate of 3 miles an hour. If A remains at Q for 15 minutes and then returns by the carriage to P, find where he will meet B.

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445. Divide

$(a-b)^2c^2+(a-b)c^3-(c^2-a^2)b^2+(c-a)b^3$ by $(a-b)c^2-(c-a)b^2$.

446. Find the H.C.F. of

$x(6x^2-8y^2)-y(3x^2-4y^2)$ and $2xy(2y-x)+4x^3-2y^3$.

447. Simplify $\dfrac{2}{a+x}-\dfrac{1}{a-x}+\dfrac{3x}{a^2-x^2}+\dfrac{ax}{a^3+x^3}$.

448. If $x=\dfrac{a-b}{a+b}$ and $y=\dfrac{a+b}{a-b}$, find the value of

$$\frac{x^2-y^2+x}{y^2-x^2+y}.$$

449. Solve the equations:

$$\frac{x^2-2\frac{1}{2}}{4}-\frac{x-3\frac{1}{2}}{5}=\frac{2x^2-3}{8}-\frac{x-5\frac{1}{2}}{3} \quad \ldots\ldots(1).$$

$$\frac{x}{\cdot 5}-\frac{1}{\cdot 05}+\frac{x}{\cdot 005}-\frac{1}{\cdot 0005}=0 \quad \ldots\ldots(2).$$

$$\left.\begin{aligned}2x+7z&=7\\ 3y+x&=2\\ 3z-4y&=x+15\end{aligned}\right\} \quad \ldots\ldots(3).$$

450. Reverse the digits of a number and it will become five-sixths of what it was before; also the difference of the two digits is 1; find the number. Also find that number of three digits which is the same when reversed, and the sum of whose digits is 16 and difference of the middle and extreme digits 2.

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451. Find the coefficient of x^4 in the product of
$x^4-ax^3+bx^2-cx+d$ and x^2+px+q.

452. Resolve into their simplest factors:

(1) $x^2-\left(a+\frac{1}{a}\right)x+1$, (2) $x^4+a^2x^2+a^4$.

453. Find the L. C. M. of
x^2-1, x^2+1, $(x-1)^2$, $(x+1)^2$, x^3-1, x^3+1.

454. Simplify $\frac{x^{3n}}{x^n-1}-\frac{x^{2n}}{x^n+1}-\frac{1}{x^n-1}+\frac{1}{x^n+1}$.

455. If $x=\sqrt{\frac{n-1}{n+1}}$, find the value of
$$\left(\frac{x}{x-1}\right)^2+\left(\frac{x}{x+1}\right)^2.$$

456. Find to four terms the square root of $1-x-x^2$.

457. Solve the simultaneous equations:
$$\frac{m}{x}-\frac{n}{y}=a, \qquad px=qy.$$

458. Two sums of money are together equal to £54. 12*s.*, and there are as many pounds in the one as shillings in the other; what are the sums?

459. A boy buys a certain number of oranges at three for 2*d.* and a third of that number at two for 1*d.*; at what price must he sell the lot to gain 20 per cent. profit? If his profit be 5*s.* 4*d.*, find the number bought.

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460. Simplify (1) $\dfrac{\frac{1}{m-n}-\frac{1}{m-p}}{\frac{1}{(m-n)^2}-\frac{1}{(m-p)^2}}$;

(2) $\left(\dfrac{x}{1-\frac{1}{x}}-x-\dfrac{1}{1-x}\right)\div\left(\dfrac{x}{1+\frac{1}{x}}+x-\dfrac{1}{1+x}\right)$.

461. Find the L. C. M. of
$9x^4-28x^2+3$, $27x^4-12x^2+1$, $27x^4+6x^2-1$, x^4-6x^2+9.

462. Extract the square root of
$$\frac{(a^2+b^2)^2}{a^4-2a^2b^2+b^4}+4\frac{a}{a+b}\times\frac{b}{a-b}.$$

463. Solve the equations:

$$\frac{a-x^2}{bx}-\frac{b-x}{c}=\frac{c-x}{b}-\frac{b-x^2}{cx} \quad \ldots\ldots(1).$$

$$\cdot 011x+\frac{\cdot 001x-\cdot 125}{\cdot 6}=\frac{5-x}{\cdot 63}-\cdot 145 \quad \ldots\ldots(2).$$

$$16\left(\frac{a-x}{a+x}\right)^3=\frac{a+x}{a-x} \quad \ldots\ldots(3).$$

$$\left.\begin{aligned} x-y-z&=-15\\ y+2z+x&=40\\ 4z-5x-6y&=-150 \end{aligned}\right\} \quad \ldots\ldots(4).$$

464. A bankrupt paid 17*s.* 6*d.* in the pound to his creditors and then gave four-fifths of what he still owed to the lawyers, and this left him £20 for current expenses; what was the amount of his debt?

465. At what time are the hands of a watch together between 5 and 6 o'clock?

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466. Divide $x^3+y^3-z^3+3xyz$ by $x+y-z$.

467. Express $(x+3a)(x+5a)(x+7a)(x+9a)$ as the difference of two square quantities.

468. Resolve $x^8+a^4x^4+a^8$ and x^8-16a^8 into their elementary factors.

469. Simplify $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}$.

470. Solve the equations:

$$\frac{x-2}{x-3}+\frac{x-3}{x-4}=\frac{x-1}{x-2}+\frac{x-4}{x-5} \quad \ldots\ldots(1).$$

$$\left.\begin{aligned} \frac{2}{x}+\frac{3}{y}&=2\\ \frac{5}{x}+\frac{10}{y}&=5\tfrac{5}{6} \end{aligned}\right\} \quad \ldots\ldots(2).$$

$$\left.\begin{aligned} x-2y+z&=0\\ 9x-8y+3z&=0\\ 2x+3y+5z&=36 \end{aligned}\right\} \quad \ldots\ldots(3).$$

471. An officer can form his men into a hollow square 5 deep, and also into a hollow square 6 deep, but the front in the latter formation contains 5 men fewer than in the former; find the number of men.

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472. If $x=b+c$, $y=c-a$, $z=a-b$; prove that

$$x^2+y^2+z^2+2yz-2zx-2xy=4b^2.$$

473. Divide

$$(ax+by)^3+(ax-by)^3-(ay-bx)^3+(ay+bx)^3$$

by $(a+b)^2x^2-3ab(x^2-y^2)$.

474. Find the H.C.F. of

$4x^3-8ax^2-20a^2x+24a^3$ and $6x^3+24ax^2+6a^2x-36a^3$.

475. Extract the square root of

$$(bc+ca+ab)^2-4abc(a+c) \quad \text{.....(1).}$$

$$x^4+2(y+z)x^3+(3y^2+2yz+3z^2)x^2+2(y^3+y^2z+yz^2+z^3)x+y^4+2y^2z^2+z^4 \quad \text{.....(2).}$$

476. Solve the equations:

$$\frac{1}{3}-\frac{7x-1}{6\frac{1}{2}-3x}=\frac{8}{3}\times\frac{x-\frac{1}{2}}{x-2} \quad \text{.....(1).}$$

$$\frac{x+2\frac{1}{2}}{15}+\frac{x+3\frac{1}{3}}{25}=\frac{x+4\frac{1}{6}}{55} \quad \text{.....(2).}$$

$$\left.\begin{aligned}(x+7)(y-3)+7&=(y+3)(x-1)+5\\5x-11y+35&=0\end{aligned}\right\} \quad \text{.....(3).}$$

477. The dimensions of a rectangular court are such that if the length were increased by 3 yards and the breadth diminished by the same, its area would be diminished by 18 square yards; and if its length were increased by 3 yards and its breadth increased by the same, its area would be increased by 60 square yards; find the dimensions.

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478. If $\dfrac{x}{(b-c)(b+c-2a)}=\dfrac{y}{(c-a)(c+a-2b)}=\dfrac{z}{(a-b)(a+b-2c)}$,

find the value of $x+y+z$.

479. Simplify the expression

$$(16x^5-20x^3+5x)^2+(1-x^2)\{16(1-x^2)^2-20(1-x^2)+5\}^2.$$

480. Reduce to its lowest terms

$$\frac{3x^3-27ax^2+78a^2x-72a^3}{2x^3+10ax^2-4a^2x-48a^3}.$$

481. Extract the square root of

$$\frac{x^4}{4}-2x^3+4x^2+\frac{ax^2}{3}-\frac{4ax}{3}+\frac{a^2}{9}.$$

482. Solve the equation:

$$x-\left(3x-\frac{2x-5}{10}\right)=\frac{1}{6}(2x-57)-\frac{5}{3}.$$

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483. Find the H.C.F. of x^5+x^4+1 and $x^6-2x^4+x^2-1$.

484. Extract the square root of

$$9x^2-24x+19-\frac{4}{x}+\frac{1}{4x^2}.$$

485. Solve the equations:

$$\tfrac{1}{3}(x+1)+\tfrac{1}{2}(x-1)-\tfrac{1}{4}(3x-7)=2 \quad \text{.....(1).}$$

$$\frac{1}{x-a}-\frac{1}{x-a+c}=\frac{1}{x-b-c}-\frac{1}{x-b} \quad \text{.....(2).}$$

$$\frac{1}{3x}-\frac{1}{7y}=\frac{2}{3}, \quad \frac{1}{2x}-\frac{1}{3y}=\frac{1}{6} \quad \text{.....(3).}$$

486. Of the candidates in a certain examination 45 per cent. passed; if there had been 30 more candidates, of whom 19 failed, the number of successful candidates would have been 44·8 per cent.; how many candidates were there?

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487. Divide $x^5+6ax^4+5a^2x^3+6a^3x^2+7a^4x+a^5$ by $x+a$.

488. Find the H.C.F. of $20a^4-3a^3b+b^4$ and $64a^4-3ab^3+5b^4$.

489. Solve the equations:

$$x-\frac{3-x}{5}=3\frac{x-1}{2}+\frac{x+1}{5}-\frac{3}{10} \quad \text{.....(1).}$$

$$\frac{1}{(a-b)(x-a)}-\frac{1}{(c-d)(x-c)}=\frac{1}{(a-b)(x-b)}-\frac{1}{(c-d)(x-d)} \quad \text{...(2).}$$

490. A tradesman sold two articles together for 46 rupees, making 10 per cent. profit on one and 20 per cent. on the other; if he had sold each article at 15 per cent. profit the result would have been the same. At what price did he sell each article?

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491. Simplify

$$(x+y+z)^2-(y-x)(z-x)-(z-y)(x-y)-(x-z)(y-z).$$

492. Find the value of

$$\tfrac{4}{5}y(y-x)-35\left[\tfrac{1}{5}(3x-4y)-\tfrac{1}{10}\left\{3x-\tfrac{5}{7}(7x-4y)\right\}\right]$$

when $x=-\frac{1}{2}$, $y=2$.

493. Extract the square root of

$$\frac{4a^2-12ab-6bc+4ca+9b^2+c^2}{4a^2+9c^2-12ca}.$$

494. Solve the equations:

$$\frac{x-8}{5}+\frac{x+4}{4}+\frac{x-1}{7}=7-\frac{23-x}{5} \quad \text{.....(1).}$$

$$\frac{x+a}{x+b}=\frac{x+3a}{x+a+b} \quad \text{.....(2).}$$

495. The express leaves Bristol at 3 P.M. and reaches London at 6 P.M.; the ordinary train leaves London at 1.30 P.M. and arrives at Bristol at 6 P.M.; if both trains travel uniformly find the time when they will meet.

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496. Divide $a^3(b-c)+b^3(c-a)+c^3(a-b)$
by $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

497. Find the H.C.F. of
$$3x^3-5x^2+5x-2 \text{ and } 2x^4-2x^3+3x^2-x+1.$$

498. Extract the square root of
$$4x^6-12x^5+13x^4-22x^3+25x^2-8x+16.$$

499. Solve the equations:
$$120x-4[5x-2\{6x+7(x-8)\}]=16-4[3x-2\{x-6(x-1)\}]\ldots(1).$$
$$\frac{x+b}{a-b}=\frac{x-b}{a+b}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$
$$\left.\begin{aligned}\frac{6}{x}+\frac{4}{y}&=3\\ \frac{9}{x}-\frac{1}{y}&=2\tfrac{3}{4}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

500. Divide the number 832 into two parts such that 30 per cent. of one part exceeds 40 per cent. of the other part by 6.

UNIVERSITY OF MADRAS.

501. Find the value of $a^3+b^3+c^3-3abc$, when $a=1$, $b=4$, $c=-5$.

502. Divide x^2+xy+y^2 by $x+\sqrt{xy}+y$.

503. Solve the equation:
$$\frac{x}{5}+\frac{x-3}{4}=\frac{x+1}{2}-\frac{x-1}{7}.$$

504. A train running from A to B meets with an accident 50 miles from A, after which it moves with three-fifths of its original velocity and arrives at B 3 hours late; had the accident happened 50 miles further on, it would have been only 2 hours late. Find the distance from A to B and the original velocity of the train.

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505. Find the numerical value of
$$\{a-(b-c)\}^2+\{b-(c-a)\}^2+\{c-(a-b)\}^2,$$
when $a=1$, $b=3$, $c=5$; and simplify the given expression.

506. Extract the square root of

$$1+2x-x^3+\tfrac{1}{4}x^4.$$

507. Solve the simultaneous equations:

$$\frac{x}{a}+\frac{y}{b}=1-\frac{x}{c},$$

$$\frac{y}{a}+\frac{x}{b}=1+\frac{y}{c}.$$

508. A number of two digits when divided by their sum gives the quotient 4; but if the digits be reversed, and the number thus formed be increased by 12 and then divided by their sum, the quotient is 8; find the number.

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509. Divide $1-\frac{x}{2}$ by $1-\frac{x}{3}-\frac{x^2}{4}$ to five terms.

510. Resolve into their elementary factors:

(1) x^6-a^6, (2) $x^2-9x+14$.

511. Solve the equations:

$$\frac{6x-a}{4x-b}=\frac{3x+b}{2x+a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$\left.\begin{aligned} x-y-z&=4\\ 3y-z-x&=16\\ 7z-x-y&=22 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

512. A can do a piece of work in 12 days, but when he has been at work for 4 days B is sent to help him, and they together finish it in 3 days; in how many days can B do the whole?

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513. Simplify the following expression:

$$\left\{\frac{(x-a)(y-b)x^7\sqrt{(xy)}}{\sqrt[3]{(by)}\sqrt[3]{(a+b)}(x-y)^7}\right\}^0 \times \frac{(x+y)(x-y)(x^2+y^2)}{x^4-y^4} \times \sqrt{(a^2+2ab+b^2)}.$$

514. Divide

$$x^4+3x^3y-x^2(y-6y^2)-3xy^2+(15x-1)y^3+5y^4 \text{ by } x^2+3xy+y^2.$$

515. In the equation

$$\frac{x+a}{a+b}+\frac{x+b}{a-b}=\frac{(a+b)^2}{a^2-b^2},$$

shew that the value of x is independent of a.

516. Two years after the Flood, when Shem had lived a sixth of his life, he begat Arphaxad; who, when he was 35 years old, begat Salah. When Salah had lived a twentieth of Shem's life he begat Eber, who after 34 years begat Peleg. Peleg begat Reu at the same age as Salah begat Eber. Reu was two years older when he begat Serug.

Serug begat Nahor at the same age as Peleg begat Reu. At a year younger Nahor begat Terah, who, when he had lived seven-sixtieths of the years of Shem's life, begat Abram. Abram was born 292 years after the Flood and lived 175 years. By how many years did Abram survive Shem, or Shem survive Abram?

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517. If $x=6$, $y=3$, $z=5$, find the numerical value of

$$\frac{x-y}{4}\times\frac{z-4}{x}+\frac{x}{y}-\frac{y}{z}.$$

518. Simplify $\sqrt[6]{\left\{\left(\frac{x^4-a^4}{x^2+a^2}\right)^3\right\}^4}$.

519. Multiply $x^4+7x^3y-8x^2y^2-13xy^3+5y^4$ by $6x^2y-3y^2x$.

520. Separate into factors

$$a^4+2a^3x-2ax^3-x^4.$$

521. Reduce to the simplest form the expression

$$\frac{4a+6b}{a+b}+\frac{6a-4b}{a-b}-\frac{4a^2+6b^2}{a^2-b^2}+\frac{4b^2-6a^2}{a^2+b^2}+\frac{20b^4}{a^4-b^4};$$

and find its value if (1) $a=b(2+\sqrt{5})$, (2) $a=2$ and $b=0$.

522. Extract the cube root of

$$\tfrac{1}{8}a^3-\tfrac{1}{4}a^2b+\tfrac{1}{6}ab^2-\tfrac{1}{27}b^3.$$

523. Solve the equations:

$$\frac{x-4}{x-2}+\frac{x+3}{x-5}=2 \quad \text{.....(1).}$$

$$\frac{x-a}{a}+\frac{x-b}{b}+\frac{x-c}{c}=1 \quad \text{.....(2).}$$

$$\frac{x}{bc}+\frac{x}{ca}+\frac{x}{ab}=a+b+c \quad \text{..... .....(3).}$$

524. In a University Examination a fourth part of the candidates and six more passed in the Second Division, a thirteenth part in the First Division, and three more than half failed; how many candidates were examined?

525. A man has in his purse sovereigns and shillings; if he receive as many sovereigns as he has in his purse and pay away his shillings and an equal number of sovereigns he will have eight coins; but if he double the number of his shillings, while retaining the original number of sovereigns, he will have nine coins. How many sovereigns and how many shillings were in his purse at first?

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526. Add together

$2a^3+4a^2+6ab$, a^2+b^2, ab^2+b^3, $3a^2b-a^3$, $ab^2-4ab-2a^2$, ab^2+a, $b-2a^2$;

and reduce the sum to its simplest form.

527. Divide $a^2b^2x^2y^2 - a^2cx^2y + 3b^3xy^2 - 3bcxy + (b^2cy - c^2)y$ by $b^2y^2 - cy$, and square the quotient.

528. If $a = \frac{1}{2}$, find the numerical value of

$$(1)\quad \sqrt[3]{(2a^3 + a^2 + a)}, \qquad (2)\quad \sqrt[3]{\left(\frac{1}{a^3} + \frac{1}{a^2} - \frac{2}{a}\right)}.$$

529. Write down the naught, first, second and fourth powers of $\sqrt{4a^2 - 4ax^2 + x^4}$. What is the square of the quantity whose square root is a?

530. Extract the square root of

$$16m^2x^2 + 36n^2y^2 + 9p^2z^2 + 36npyz + 24pmzx + 48mnxy.$$

531. Solve the equations:

$$\frac{bx}{c} + \frac{cx}{a} + \frac{ax}{b} = \frac{(bc^2 + ca^2 + ab^2)^2}{a^2b^2c^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$2x + 3y = 5x - \tfrac{1}{2} = 4y + \tfrac{5}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

532. Two trains start at the same time from A and B for the junction C. The train from A should run at 24 miles an hour and reach the junction half-an-hour before that from B, which travels 18 miles an hour; but the former is so retarded as only to run at an average rate of 22 miles an hour, and so the two trains arrive at the junction at the same time. How far are A and B respectively from C, and how long are the trains upon the road?

533. A and B went out shooting; A shot three pheasants for every five partridges, B five pheasants for every nine partridges, and A shot four birds to B's five; how many pheasants and how many partridges had they brought down when they had shot 126 birds?

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534. Simplify

$$(a+b+c)(a-b+c) - \{(a+c)^2 - b^2 - (a^2 + b^2 + c^2)\}.$$

535. Multiply $x^2 + 3x + 9$ by $x^2 - 3x + 9$ and $a^2 + ab + b^2$ by $a + b$.

536. Divide $x^3 - 7x - 6$ by $x - 3$ and $\frac{1}{a^3} + \frac{1}{b^3}$ by $\frac{1}{a} + \frac{1}{b}$.

537. Find the square root of

$$x^6 - 4x^4 + 14x^3 + 4x^2 - 28x + 49 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$a + b + c - 2\sqrt{bc} - 2\sqrt{ca} + 2\sqrt{ab} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

538. Solve the equations:

$$\frac{7x+5}{3} - \frac{24-4x}{5} + 5 = \frac{3x+7}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\frac{x-bc}{a} + \frac{x-ca}{b} + \frac{x-ab}{c} = 2(a+b+c) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$$\left.\begin{aligned} ax+by&=c\\ a_1x+b_1y&=c_1\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

$$\left.\begin{aligned} 3x+2y&=1{\cdot}4\\ 5x-{\cdot}1y&=1{\cdot}99\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

$$\frac{1}{y}+\frac{1}{z}=4, \quad \frac{1}{z}+\frac{1}{x}=5, \quad \frac{1}{x}+\frac{1}{y}=3 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

539. A merchant goes to three bazars in succession; at the first he gains 15 per cent. on his capital, at the second 20 per cent. on this increased capital, and at the third 25 per cent. on what he then possessed, and on his return home he finds that he has gained Rs. 2639; what was his original capital?

540. A certain fraction becomes $\frac{1}{2}$ when 1 is added to its denominator, and $\frac{1}{3}$ when 2 is taken away from its numerator; what is the fraction?

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541. If $a=1$, $b=2$, $c=3$, $d=0$; find the value of

$$\frac{a^2b+b^2c+c^2d+d^2a}{(a+b)(c+d)-\{(a-d)+(c-b)\}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\sqrt[n]{b-a}+\sqrt[3]{4(c-a)}-\sqrt[4]{3(8a+5b+3c-2d)} \quad \ldots\ldots\ldots\ldots\ldots(2).$$

542. Simplify the following expression:

$$\{2x^2-y(3x-y)+x+(y-1)\}-\{2(x+y)+3(y+1)(y-1)+2x(x-2y)\}$$
$$+[(x-y+1)-\{xy-2(y^2+y-2)+2\}].$$

543. Multiply $\frac{1}{a^3}-\frac{4b}{a^2}+\frac{4b^2}{a}-b^3$ by $\frac{1}{a^2}-\frac{2b}{a}+b^2$.

544. Without removing the brackets divide

$(a^2-b^2)x^2-2(a^2+b^2)xy+(a^2-b^2)y^2$ by $(a+b)x-(a-b)y$.

545. Find the H.C.F. of

$2a^4-3a^3b-a^2b^2-2ab^3$ and $a^5-3a^4b+4a^3b^2-5a^2b^3+3ab^4-2b^5$.

546. Reduce to their simplest forms

(1) $\frac{x^2+x-12}{x^2+7x+12}$; (2) $\frac{1}{a+x}+\frac{1}{a-x}+\frac{2a}{a^2+x^2}$;

(3) $\dfrac{\frac{a}{a+b}+\frac{b}{a-b}}{\frac{a}{a-b}-\frac{b}{a+b}}$.

547. Extract the square root of

$$a^2+4b+9\sqrt[3]{c^2}+4\sqrt{a^2b}-6a\sqrt[3]{c}-12\sqrt[6]{b^3c^2}.$$

548. Solve the equations:

$$\frac{x^2-a}{x-\sqrt{a}}+\frac{x^2-a}{x+\sqrt{a}}=x-\frac{x-\sqrt{a}}{2} \quad \ldots\ldots(1).$$

$$\left.\begin{aligned} 2x-a-\frac{y-b}{a+b}&=a \\ 2y-b+\frac{x-a}{b}&=2b-y \end{aligned}\right\} \quad \ldots\ldots(2).$$

$$xyz=xy+zx-yz=4\,(yz+xy-zx)=6\,(zx+yz-xy) \quad \ldots\ldots(3).$$

549. A merchant speculated during twenty voyages; on examining his accounts he found that, on an average, in each prosperous voyage he had gained a sum equal to $\frac{1}{6}$ of his original capital, and that in each adverse voyage he had lost a sum equal to $\frac{1}{4}$ of the same. On the whole however his capital was increased by $\frac{5}{12}$ of itself; how many prosperous voyages did he make?

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550. Prove that

$$(a^2+ab+b^2)\,(c^2+cd+d^2)$$
$$=(ac+ad+bd)^2+(ac+ad+bd)\,(bc-ad)+(bc-ad)^2.$$

551. Divide (1) $2x^3-9ax^2+11a^2x-6a^3$ by $x-3a$;
(2) $8x^3-8x^2y+4xy^2-y^3$ by $2x-y$.

552. Find the H. C. F. of $4x^4+9x^3+2x^2-2x-4$ and $3x^3+5x^2-x+2$; and the L. C. M. of $3x^2-5x+2$ and $4x^3-4x^2-x+1$.

553. Simplify the following fractions:

$$\frac{3a^4-a^2b^2-2b^4}{10a^4+15a^3b-10a^2b^2-15ab^3} \quad \ldots\ldots(1).$$

$$\frac{1}{a-\sqrt{a^2-x^2}}-\frac{1}{a+\sqrt{a^2-x^2}} \quad \ldots\ldots(2).$$

554. Find the square root of

$$\frac{1}{a^4}+\frac{1}{b^4}-\frac{1}{a^2b^2}+\frac{2}{a^3b}-\frac{2}{ab^3}.$$

555. Solve the equations:

$$\frac{6x+18}{13}-4\tfrac{5}{6}-\frac{11-3x}{36}=5x-48-\frac{13-x}{12}-\frac{21-2x}{18} \quad \ldots\ldots(1).$$

$$\frac{3-3x}{2}-\frac{2-2x}{3}=\frac{1-x}{4} \quad \ldots\ldots(2).$$

$$\left.\begin{aligned} 5x+2y+z&=30 \\ 2x+5y+10z&=129 \\ \frac{x}{2}+\frac{4y}{5}-\frac{z}{10}&=4 \end{aligned}\right\} \quad \ldots\ldots(3).$$

556. A and B start together on a certain journey, and when they have walked together a distance of a miles A finds it necessary to return home and walks at twice his former rate. He then starts again at $\frac{m}{n}$ times his original pace and just at the end of the journey overtakes B, who since A left him had gone at $\frac{n}{m}$ times his original pace; how long was the journey?

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557. Find the continued product of

$$x+\sqrt{3x}+3,\ x^2-3x+9 \text{ and } x-\sqrt{3x}+3.$$

558. Resolve each of the following expressions into three factors:

$$b^3-a^3-(c^2-ab)(b-a) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$a(b^2+c^2-a^2)+b(a^2+c^2-b^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

559. Find the value of $\frac{(x-a)(x-b)}{(x-a-b)^2}$, when $x=\frac{a^2+ab+b^2}{a+b}$.

560. Find the L.C.M. of $x^3-7x-36$, $x^3-13x-12$, x^2+4x+3.

561. Shew that

$$\left(\frac{1}{a}-\frac{1}{c}\right)^2+\frac{4}{(a+c)^2}=\left(\frac{a+c}{ac}-\frac{2}{a+c}\right)^2.$$

562. Simplify the expressions:

$$\frac{a^2}{(x-a)^n}+\frac{2a}{(x-a)^{n-1}}+\frac{1}{(x-a)^{n-2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\frac{a+b-c}{c-a}-\frac{a+b+c}{c-b}-\frac{(a+b+c)(a-b)}{c^2-(a+b)c+ab} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

563. Solve the equations:

$$\frac{3x+5}{x+1}=\frac{4x+8}{3x+3}+\frac{10x+1}{6x+3} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\sqrt{x+9}-\sqrt{x-8}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$$\left.\begin{aligned} ax+by+cz&=a+b+c \\ \frac{ax}{b+c}+\frac{by}{a+c}&=1 \\ \frac{2x}{b+c}+\frac{2y}{a+c}&=\frac{1}{a}+\frac{1}{b} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

564. A, B and C work together at building a wall for ten days, after which B stops working and A and C together finish it in five days; if A and B together can do as much in a day as C can in three days, and three days' work of B is equal to four days' work of C, find the time in which each could separately build the wall.

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565. Reduce to its simplest form

$$\frac{x^3-1}{x^3+1}+\frac{x^3+1}{x^3-1}-\frac{2x^4}{x^4+x^2+1}.$$

566. Divide

$$(a+b)(b+c)x^5+\{(a+b)^2+(b+c)^2\}x^4$$
$$+(a-b)(b+c)x^2+(a+b)(a+c)x-(a+c)(b+c)$$

by

$$(b+c)x^2+(a+b)x-(b+c).$$

567. Find the H.C.F. of x^5-2x^4-x-1 and x^5+2x^3+x+1.

568. If $\frac{a}{b}+\frac{c}{d}=\frac{b}{a}+\frac{d}{c}$, prove that

$$\frac{a^3}{b^3}+\frac{c^3}{d^3}=\frac{b^3}{a^3}+\frac{d^3}{c^3}.$$

569. Extract the square root of

$$a^4+\frac{1}{a^4}+a^2+\frac{1}{a^2}+2\left\{a^3-\frac{1}{a^3}-\left(a-\frac{1}{a}\right)\right\}.$$

570. Solve the equations:

$$\frac{x-1}{3}-\frac{1\frac{1}{2}-2x}{4\frac{1}{2}-5x}=\frac{\frac{1}{3}(2x-1)}{2}+2-\frac{7}{18x+6}-\frac{7x}{6x+2} \quad \text{.....(1).}$$

$$\left.\begin{aligned} y+z&=x+1\\ z+x&=3y+1\\ x+y&=2(z+1)\end{aligned}\right\} \quad \text{.....(2).}$$

571. A, B, C start at the same instant from P to run to Q, their rates being such that B is always as much behind A as he is in advance of C. After A has reached Q he returns at once to P at the same rate and meets B at a point whose distance from Q is one-fourth of PQ; shew that A meets C at a distance from P which is one-third of PQ.

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572. Without removing the brackets divide

$$(m+n)a^4+(m^2+n^2+2mn-1)a^3$$
$$-(m+n)(1-2mn)a^2+(m+n-2mn)a-1$$

by

$$(m+n)a-1.$$

573. If $x+\frac{1}{x}=2a$, $x-\frac{1}{x}=2b$, $y+\frac{1}{y}=2c$, $y-\frac{1}{y}=2d$; find the value of $xy+\frac{1}{xy}$.

574. Find the H.C.F. of x^4-4x^2+5x-4 and x^4-x^3+x-1; and the L.C.M. of x^4+4x^2-5, x^2-4x+3 and $2x^3+x^2-8x+5$.

575. If $ap=bq=cr$, shew that

$$\frac{p^2}{qr}+\frac{q^2}{rp}+\frac{r^2}{pq}=\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}.$$

576. Simplify

$$\frac{ab}{a+b}\left(3c+\frac{b}{a}\right)-\frac{b^2}{(a+b)^3}(a^2+b^2)-2a\left(\frac{b}{a+b}\right)^3 \quad \ldots\ldots(1).$$

$$\frac{\dfrac{(a+b)^2+(a-b)^2}{b-a}-(a+b)}{\dfrac{1}{b-a}-\dfrac{1}{a+b}} \div \frac{(a+b)^3+(b-a)^3}{(a+b)^2-(a-b)^2} \quad \ldots\ldots(2).$$

577. Solve the equations:

$$\frac{6-5x}{5}-\tfrac{3}{14}\times\frac{7-2x^2}{x-1}-1\tfrac{1}{10}=\frac{1+3x}{7}-x+\tfrac{1}{35} \quad \ldots\ldots(1).$$

$$\frac{7abc}{a+b+c}+\frac{a^2+b^2+c^2}{(a+b+c)^2}x+\frac{2abc\,(bc+ca+ab)}{(a+b+c)^3}=8x \quad \ldots\ldots(2).$$

$$\left.\begin{aligned}(a-b)\,x+(b-c)\,y+cz&=1\\ 2ax+by+\tfrac{2}{3}cz&=2\\ (a+b-c)\,x+(a-2b+2c)\,y+2bz&=3\end{aligned}\right\} \quad \ldots\ldots(3).$$

578. A alone can do a piece of work in a hours, A and C together can do it in b hours, and C's work is $\frac{1}{n}$th of B's; the work has to be completed in c hours. Find (1) how long after A has commenced B and C should relieve him so as to finish the work in time; (2) how long after A has commenced B and C should join him so that the three working together may just complete the work in time.

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579. Divide $bc\,(b-c)+ca\,(c-a)+ab\,(a-b)$ by $(a-b)\,(a-c)$.

580. Resolve the first of the following expressions into two factors and the second into four factors:

$$x^4+x^2+1 \quad \ldots\ldots(1).$$

$$a\,(b^3-c^3)-bc\,(b^2-c^2)-a^3\,(b-c) \quad \ldots\ldots(2).$$

581. In what case is x^m+a^m divisible by $x+a$? Shew that the last digit in $3^{2n+1}+2^{2n+1}$ must be 5 if n be any positive integer.

582. Find the H.C.F. of x^3-7x+6 and $6x^3-7x^2+1$.

583. Simplify the following expressions:

$$\frac{1}{x^2+x+1}+\frac{2x}{x^4+x^2+1} \quad \ldots\ldots(1).$$

$$\frac{b+c-2a}{b-a}-\frac{(b-c)^2}{(a-b)\,(a-c)}+\frac{b+c-2a}{c-a} \quad \ldots\ldots(2).$$

584. If $x=\dfrac{2ac}{a+c}$, prove that the value of

$$\frac{(x-a)^2+(x-c)^2}{a^2+c^2}+\frac{4ac}{(a+c)^2}$$

is the same for all values of a and c.

585. Solve the equations:

$$\frac{1-4x}{5}+\frac{5}{9}-\frac{3-7x}{4}=x \quad \text{.....(1).}$$

$$\left.\begin{aligned} ax-by&=a-b\\ \frac{x}{2a}+\frac{y}{2b}&=\frac{1}{a+b}\end{aligned}\right\} \quad \text{.....(2).}$$

$$\left.\begin{aligned} \sqrt{y}-\sqrt{y-2x}&=\sqrt{48-2x}\\ y(x-15)&=36\end{aligned}\right\} \quad \text{.....(3).}$$

586. A room of which the floor is rectangular is such that the addition of a foot to the height would increase the area of the walls as much as the addition of a foot to both the length and breadth, the increase in each case being 60 square feet; and if the floor were made square, the perimeter remaining the same as before, its area would be increased by 9 square feet; find the length, breadth and height of the room.

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587. Simplify

$$\{\sqrt{(1-a^2)(1-x^2)}+ax\}^2-2ax\{\sqrt{(1-a^2)(1-x^2)}+ax\}+x^2.$$

588. Divide $x^{10}-a^5x^5+a^{10}$ by x^2-ax+a^2.

589. Shew that b^2-ac is a factor of

$$(2b^2+a^2-ac)(2b^2+c^2-ac)-b^2(a+c)^2;$$

and that the other factor is positive for all values of a, b and c.

590. Prove that the two expressions

$$(a^2-a+1)(b-c)+(b^2-b+1)(c-a)+(c^2-c+1)(a-b)$$

and $(a^2-a+1)(b^2-c^2)+(b^2-b+1)(c^2-a^2)+(c^2-c+1)(a^2-b^2)$

are equal.

591. Shew that $\dfrac{2x^3+5x^2+19}{x^3+5x+12}$ is in its lowest terms.

592. If $\dfrac{a-1}{x}-\dfrac{a-2}{y}=\dfrac{1}{b}$ and $\dfrac{b-1}{x}-\dfrac{b-2}{y}=\dfrac{1}{a}$, shew that

$$\frac{c-1}{x}-\frac{c-2}{y}=\frac{c}{ab}.$$

593. Simplify the following:

$$\frac{1+\frac{x}{2a}}{\left(1+\frac{x}{a}\right)^2}+\frac{1+\frac{3a}{2x}}{\left(1+\frac{a}{x}\right)^2} \quad \text{.....(1).}$$

$$\frac{1}{(x+2)(x+4)}+\frac{1}{(x+1)(x+3)}-\frac{3}{(x+1)(x+2)(x+3)(x+4)} \quad \text{...(2).}$$

594. Arrange the expression

$$(b+c)^2(a+d)^2-4(ab+cd)(ac+bd)$$

according to powers of b, and hence find its square root.

595. Solve the equations:

$$\frac{12x+19}{18}-\frac{7x-2}{3x-10}=\frac{8x-25}{12}+\frac{5}{36} \quad \text{.....(1).}$$

$$\left.\begin{aligned} x+y+z&=-(b-c)(c-a)(a-b)\\ ax+by+cz&=0\\ a^2x+b^2y+c^2z&=0 \end{aligned}\right\} \quad \text{.....(2).}$$

596. A horseman travelling at a walking pace of 4 miles an hour meets a bandy going in the opposite direction at the rate of 2 miles an hour; after proceeding at the same pace for half-an-hour he turns and canters back until he overtakes the bandy. If he had continued for another quarter-of-an-hour before turning, the bandy would have been seven-eighths of a mile further on before it was overtaken; find the rate at which the horseman cantered.

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597. Divide the difference between

$(x+a)(x+b)(x+c)$ and $(y+a)(y+b)(y+c)$ by $x-y$.

598. Find the H.C.F. of x^4-x^3+7x+5 and x^4-x^2+2x+2.

599. Simplify

(1) $\dfrac{ac-1}{(a-b)(1+ax)}+\dfrac{bc-1}{(b-a)(1+bx)}$; (2) $\dfrac{2^{2n+1}-2^{n+2}+2}{2^{2n+1}-2^{n+1}}$.

600. If $x=\dfrac{(2c-b)^2-ab}{a+3b-4c}$, evaluate $\sqrt{\dfrac{x+a}{x+b}}$.

601. Prove that

$$\left\{\frac{(mn+1)^2}{m^2n^2}-\frac{4}{mn}\right\}y^2+2\left\{\frac{(mn+1)^2}{mn}-2mn-\frac{2}{mn}\right\}y+(mn+1)^2-4mn$$

is a perfect square, and find its square root.

602. If $x^2+y^2=1$, $2xy=\dfrac{a-b}{a+b}$, $2nx=x+y$; shew that $\left(1-\dfrac{1}{n}\right)^2=\dfrac{b}{a}$.

603. Solve the equations:

$$\frac{1}{x-3}-\frac{1}{x-4}=\frac{3}{3x-13}-\frac{3}{3x-16} \quad \ldots\ldots(1).$$

$$\left.\begin{aligned} ax-by&=\frac{b-a}{2}\\ by+ax&=c(1+z)\\ cz-by&=\frac{b-c}{2}\end{aligned}\right\} \quad \ldots\ldots(2).$$

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604. If $x=\dfrac{b^2+c^2-a^2}{2bc}$, $y=\dfrac{c^2+a^2-b^2}{2ca}$, $z=\dfrac{a^2+b^2-c^2}{2ab}$;

find in the simplest forms the values of

$$(b+c)\,x+(c+a)\,y+(a+b)\,z \quad \ldots\ldots(1),$$

$$\frac{x+yz}{y+xz} \quad \ldots\ldots(2).$$

605. Simplify

$$\frac{x^4-(x-1)^2}{(x^2+1)^2-x^2}+\frac{x^2-(x^2-1)^2}{x^2(x+1)^2-1}+\frac{x^2(x-1)^2-1}{x^4-(x+1)^2} \quad \ldots\ldots(1).$$

$$\frac{x-1}{x^2+x+1}+\frac{x+1}{x^2-x+1}-\frac{2x(x^2-2)}{x^6-1} \quad \ldots\ldots(2).$$

$$\frac{1+\dfrac{a-bx}{c+bx}}{\dfrac{x-2(a+c)}{1+\dfrac{x}{a+c-x}}}-\frac{1+\dfrac{a-bx}{c+bx}}{\dfrac{x-2(a+c)}{1+\dfrac{x}{a+c-x}}} \quad \ldots\ldots(3).$$

606. If the two expressions

$$ax^3-c(3a+b)\,x^2+(a^3+bc^2)\,x+d,$$

and

$$bx^3+c(a-b)\,x^2+a(c^2-a^2)\,x-d,$$

have a common quadratic factor, prove that this factor is an exact square.

607. There are two quantities a, b, of which the L.C.M. is x and the H.C.F. is y; if $x+y=ma+\dfrac{b}{m}$, prove that

$$x^3+y^3=m^3a^3+\frac{b^3}{m^3}.$$

608. Find the H.C.F. of

$$x^8+36x^3-4x^2+8x-17 \text{ and } x^6+8x^3-x^2+2x-4.$$

609. If $bc+ca+ab=1$, prove that

$$\left\{1-\frac{a^2}{1+a^2}-\frac{b^2}{1+b^2}-\frac{c^2}{1+c^2}\right\}^2=\frac{4a^2b^2c^2}{(1+a^2)(1+b^2)(1+c^2)}.$$

610. Extract the square root of

$$x^2-2+\frac{4x^3+9x^2+4x+9}{x^4+4x^3+5x^2+4x+4}.$$

611. Solve the equations:

$$\frac{10x+47}{18}-\frac{12x+38}{13x+23}=\frac{5x+11}{9} \quad \text{.....(1).}$$

$$\frac{x+2y+1}{2}=\frac{y+3z+2}{3}=\frac{z+4x+3}{4}=2 \quad \text{.....(2).}$$

612. A and B play four games of chance, of which A wins the first and last and B the other two. The amount which each stakes for the first game is half the whole sum of money possessed by both together, and for the other games half the money possessed by the loser of the preceding game. At the end of the fourth game A finds that he has 18 shillings less than he would have had if he had won them all, and B finds that he has 9 shillings less than he had at starting; find the amount of money possessed by each at first.

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613. Divide the difference of

$$(x+a-b)(x^2-bx+b^2),$$

and

$$(x-a+b)(x^2-ax+a^2),$$

by $a-b$.

614. For what values of n is x^n+a^n divisible by $x+a$, and x^n-a^n by $x-a$? Shew (1) that $4\times 2^{4n}+2^{2n+2}+1$ is divisible by 9; (2) that $(1-x)^{2n}-(4-7x-x^2)^n$ is divisible both by $x+3$ and $2x-1$.

615. Simplify (1) $\dfrac{3}{2(1+x)}-\dfrac{5}{2(1-x)}+\dfrac{6}{1-2x}$;

(2) $\dfrac{bc-a^2}{(b^2-ac)(ax-b)}+\dfrac{c^2-ab}{(ac-b^2)(bx-c)}$.

Shew that the numerator of (1) in its simplest form is the sum of two squares.

616. Find the H.C.F. of $6x^3-7x^2+1$ and $3x^4-8x^3-12x^2+1$.

617. Prove that $\dfrac{(a^2+b^2)(1+a^2b^2)}{a^2b^2}$ is the sum of two squares.

618. Solve the equations:

$$\frac{x-3}{x-5}-\frac{x-7}{x-9}=\frac{x-11}{x-13}-\frac{x-15}{x-17} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\sqrt{a-bx}-\sqrt{b-ax}=(\sqrt{a}-\sqrt{b})\sqrt{1-x} \quad \ldots\ldots\ldots\ldots\ldots(2).$$

$$\left.\begin{aligned} ax-by&=\tfrac{3}{2}(b-a)\\ by+2cz&=\tfrac{3}{2}(a-c)\\ ax-by+cz&=0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

619. A and B start at the same time from two places C and D and meet at a point 10 miles nearer D than C. If each had travelled a mile an hour faster they would have met an hour sooner; and if A had travelled half a mile an hour slower and B half a mile an hour faster, they would have met $2\frac{1}{2}$ miles further from D; find the rate of travelling of each and the distance between C and D.

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620. Divide $a(a+b)(a+c)-b(b+c)(b+a)$ by $a-b$ and by $a+b+c$.

621. Assuming that x^n-y^n is divisible by $x-y$, shew that

$$(ab)^n-(bc)^n+(cd)^n-(da)^n$$

is divisible by $ab-bc+cd-da$.

622. Find the H.C.F. of a^5+2a^2+4 and $a^5-3a^4+2a^3-4$.

623. Simplify:

$$\frac{(n+1)^2(n+2)^2-(n-1)^2(n-2)^2}{(n+1)^3+n^3+(n-1)^3} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

$$\frac{1}{(x+1)(2x+1)}+\frac{1}{(2x+1)(3x+1)}+\frac{1}{(3x+1)(4x+1)}+\frac{1}{(4x+1)(5x+1)} \quad \ldots\ldots(2).$$

624. Shew that

$$(a-b)^2(c+d)^2+4ab(c^2+d^2)-4cd(a^2+b^2)$$

is an exact square.

625. Prove that

$$\frac{(a+b)^3-(b+c)^3+(c+d)^3-(d+a)^3}{(a+b)^2-(b+c)^2+(c+d)^2-(d+a)^2}=\frac{3}{2}(a+b+c+d).$$

626. If $\frac{1}{x}=\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ shew that

$$\frac{bc}{(x-b)(x-c)}+\frac{ca}{(x-c)(x-a)}+\frac{ab}{(x-a)(x-b)}=0.$$

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627. Solve the equations:

$$\frac{3}{x-3}-\frac{4}{x+9}-\frac{5}{x-27}+\frac{6}{x-15}=0 \quad \text{.....(1).}$$

$$\sqrt{a-m^2x}-\sqrt{b-n^2x}=\sqrt{a+b-(m+n)^2x} \quad \text{.....(2).}$$

$$\frac{a}{y}+\frac{b}{z}=\frac{a}{z}+\frac{b}{x}=\frac{a}{x}+\frac{b}{y}=c \quad \text{.....(3).}$$

628. A gentleman went out for a walk, and after having been out twelve minutes was overtaken by his servant, who had run from the house at thrice his master's pace; the master then bade the servant run back at the same rate to the house and bring his cigars, while he walked on at his former pace. If the master was one mile from the house when overtaken the second time, at what rate did he walk?

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629. Divide $\frac{b^3}{c^3}+\frac{c^3}{a^3}+\frac{a^3}{b^3}-3$ by $\frac{b}{c}+\frac{c}{a}+\frac{a}{b}$.

630. Find without multiplication the product of $a^4+a^3+a^2+a+1$ and $a^4-a^3+a^2-a+1$.

631. From $(x+b-c)(x+c-a)(x+a-b)$
subtract $(x-b+c)(x-c+a)(x-a+b)$,
and divide the remainder by $(b-c)(c-a)(a-b)$.

632. Find the H.C.F. of $a^5+29a-15$ and $2a^5-3a^4+16a^2+a-10$.

633. Simplify:

$$\frac{a^2}{x(a-x)}+\frac{x^2}{a(x-a)}-\frac{(a-x)^2}{ax} \quad \text{.....(1).}$$

$$\frac{(ac+bd)^3-(ad+bc)^3}{(a-b)(c-d)}-\frac{(ac+bd)^3+(ad+bc)^3}{(a+b)(c+d)} \quad \text{.....(2).}$$

634. If $x=\frac{a+1}{a-1}$, $y=\frac{b+1}{b-1}$, $z=\frac{c+1}{c-1}$; prove that

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{(yz+1)(zx+1)(xy+1)}=\frac{(a^2+1)(b^2+1)(c^2+1)}{(bc+1)(ca+1)(ab+1)}.$$

635. Extract the square root of

$$(a^2+b^2)(a^2b^2+1)-2ab(a^2-1)(b^2-1)-4a^2b^2.$$

636. Solve the equations:

$$\frac{x-4}{(x-1)(x-3)}+\frac{x-7}{(x-6)(x-1)}+\frac{x-9}{(x-3)(x-6)}=\frac{3}{x} \quad \text{.....(1).}$$

$$\sqrt{15-9x}+\sqrt{10-4x}=\sqrt{5-x} \quad \text{.....(2).}$$

$$\left.\begin{aligned} 4ax+(a+1)y&=(3a-1)z \\ (3a-1)(1+y-x)&=z \\ x+y&=z \end{aligned}\right\} \quad \text{.....(3).}$$

637. A man rowing against a stream meets a log of wood which is being carried down by the current. He continues rowing in the same direction for a quarter of an hour longer and then turns and rows down the stream, overtaking the log $1\frac{1}{2}$ miles lower down than the point where he first met it; at what rate does the current flow?

---

638. Remove brackets from

$$c-a-b-\{(b+c-a)-(a-b-c)-(c+a-b)\}.$$

639. If $x=\dfrac{b-c}{m-a}$, $y=\dfrac{c-a}{m-b}$, $z=\dfrac{a-b}{m-c}$; find the value of

$$x+y+z+xyz.$$

640. Prove that $2^{4n}-1$ is divisible by 15.

641. Reduce to its lowest terms

$$\frac{20a^5+11a^3+24}{24a^5+11a^2+20}.$$

642. Find the L.C.M. of $1+a+a^5$ and $1+a^4+a^5$.

643. Simplify

$$\frac{\dfrac{a+b+c}{a+b-c}+\dfrac{c+a-b}{b+c-a}}{\dfrac{a+b-c}{c+a-b}+\dfrac{b+c-a}{a+b+c}} \div \frac{\dfrac{a+b+c}{a+b-c}}{\dfrac{b+c-a}{c+a-b}}.$$

644. Find what term is wanting to make the expression

$$a^2x^4+64b^2-4(ax^2+8b)(a-b)x$$

a complete square.

645. Solve the equations:

$$\frac{x-7}{x-3}+\frac{x-2}{x-9}+\frac{x-4}{x-1}=3 \quad \ldots\ldots(1).$$

$$\sqrt{\frac{x-a}{x-b}}+\frac{a}{x}=\sqrt{\frac{x-b}{x-a}}+\frac{b}{x} \quad \ldots\ldots(2).$$

$$\left.\begin{aligned}(a^2+b^2)(x-1)&=ab(2x-y)\\ 4x&=y+2\end{aligned}\right\} \quad \ldots\ldots(3).$$

646. A person sets out to walk to a certain town, but when he has accomplished a quarter of his journey he finds that, if he continues at the same pace, he will have gone only five-sixths of the whole distance when he ought to be at his destination, and he therefore increases his speed by a mile an hour and arrives just in time; find his rates of walking.

---

647. Divide $(a-b)(a+b-c)+(b-c)(b+c-a)$ by $a-c$.

648. Separate into four factors:

$$(a^2-b^2)^2+(c^2-d^2)^2-(a+b)^2(c-d)^2-(a-b)^2(c+d)^2.$$

649. Shew (1) that $(a-1)a^n+(b-1)b^n$ is not divisible by $a+b$; and (2) that $(2a+b)^n-(a+2b)^n-a^n+b^n$ is divisible both by $a+b$ and $a-b$, whether n be even or odd.

650. If $x^2-(2q-1)x+2q$ and $x^2-2qx+2(q+1)$ have a common factor, determine q; and hence find the L.C.M. of the two expressions.

651. Simplify

$$(a+b+c)(l+m+n)+(b+c-a)(m+n-l)+(c+a-b)(n+l-m)$$
$$+(a+b-c)(l+m-n) \quad \ldots\ldots\ldots\ldots(1).$$

$$\frac{a-b}{(x+a)(x+b)}+\frac{b-c}{(x+b)(x+c)}+\frac{c-d}{(x+c)(x+d)}+\frac{d-a}{(x+d)(x+a)} \quad \ldots(2).$$

652. Express the following as the difference of two squares:

$$\{(a+b)x-(c+d)y\}\{(a-b)x-(c-d)y\}.$$

653. If $(m+a)(b+c)+a^2=(m+b)(c+a)+b^2=(m+c)(a+b)+c^2=n$, determine the values of m and n in terms of a, b, c.

654. Find the cube root of

$$(x+y)^6-(x-y)^6-12xy(x^2-y^2)^2.$$

655. Solve the equations:

$$\frac{a-b}{x+c}-\frac{a-b}{x+d}=\frac{c-d}{x+a}-\frac{c-d}{x+b} \quad \ldots\ldots\ldots\ldots(1).$$

$$\sqrt{ax+c}+\sqrt{ax+d}=\sqrt{bx+c+e}+\sqrt{bx+d+e} \quad \ldots\ldots\ldots(2).$$

$$\left.\begin{aligned} a(x+y)+b(x-y)&=a^2-ab+b^2\\ a(x+y)-b(x-y)&=a^2+ab+b^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(3).$$

656. A letter-carrier has a hours allowed him for going from A to B and back again, including c hours for rest at B, but he finds that he can get b hours for rest by going d miles an hour faster each way; find his ordinary speed and the distance from A to B.

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657. Resolve into factors

$$(2a+2b-ab)^2-(b^2-4a)(a^2-4b).$$

658. Simplify

$$\frac{b+c}{2bc}(b^2+c^2-a^2)+\frac{c+a}{2ca}(c^2+a^2-b^2)+\frac{a+b}{2ab}(a^2+b^2-c^2);$$

and shew that if $a+b+c=0$, then

$$\frac{a^2+b^2+c^2}{a^3+b^3+c^3}+\frac{2}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=0.$$

659. Find the H.C.F. of $x^4-39x-22$ and $11x^4-39x^3-8$.

660. Extract the square root of

$$(x-1)(x^3+4)+\left(\frac{x}{2}+\frac{2}{x}\right)^2.$$

661. A man being asked his age replied: "Ten years ago I was five times as old as my son, but twenty years hence I shall be only twice as old;" what was his age?

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662. Simplify

$$\frac{x^3+3x^2+5x+15}{x^3+2x^2+5x+10}+\frac{x^4+x^3+3x^2+x-2}{x^4+2x^3+3x^2+4x-4}.$$

663. Shew that

$$(x^3-3x)^4-8(x^6-6x^4+9x^2-2)$$

is an exact square; and resolve the expression into factors.

664. Shew that (1) if $a+b+c=0$, then

$$a(b-c)^3+b(c-a)^3+c(a-b)^3=0;$$

and (2) if $a+b+c=1$, $bc+ca+ab=\frac{1}{3}$, $abc=\frac{1}{27}$,

then $$\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}=\frac{27}{4}.$$

665. Solve the equations:

$$4\frac{x+5}{x+6}+3\frac{x+6}{x+5}=7 \quad \ldots\ldots(1).$$

$$\left.\begin{aligned} x+y+z=ax+by+cz=0 \\ \frac{x}{b-c}+\frac{y}{a-c}+\frac{z}{a-b}=1 \end{aligned}\right\} \quad \ldots\ldots(2).$$

---

666. Simplify

$$\frac{(a+b)^2}{(x-a)(x+a+b)}-\frac{x+a+2b}{2(x-a)}+\frac{(a+b)x}{x^2+bx-a^2-ab}+\frac{1}{2}.$$

667. Transform

$$(x^2+y^2+z^2+2xy)^2-2(x+y)^2z^2$$

into the sum of two perfect squares.

668. Solve the equations:

$$\frac{x+7}{4}+\frac{2x-4}{7}=\frac{3x-12}{5}+3 \quad \ldots\ldots(1).$$

$$\left.\begin{aligned} bx+cy+az=b+c \\ cx+ay+bz=c+a \\ ax+by+cz=a+b \end{aligned}\right\} \quad \ldots\ldots(2).$$

669. Three equal vessels A, B, C are placed one above another, A being the highest; A is full, B half full and C empty. In the bottom of A is a hole which would empty it in 16 minutes, in the bottom of B a hole which would empty it in 4 minutes; how long will it be before C is full?

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670. Multiply $a^2+2b^2+9c^2-9bc+6ca-3ab$ by $a+2b-3c$ and divide the product by $a-b+3c$.

671. Simplify

$$\frac{(a^4-b^4)^2+2a^6b^2+5a^4b^4+2a^2b^6}{(a^2+ab+b^2)^2(a^2-ab+b^2)^2}.$$

672. If $x=11$, find the value of

$$\sqrt[3]{[(x+2)\sqrt{x-2}-2\{\sqrt[3]{11x^2}-x+2\sqrt{x-2}\}]}.$$

673. Extract the square root of

$$x^4-4x^3+6x^2-4x+1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$x^2+4x+10+\frac{12}{x}+\frac{9}{x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

674. Solve the equations:

$$\frac{x+3}{8}-\frac{x-3}{10}=\frac{x+5}{6}-\frac{x-7}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\tfrac{1}{3}\left(\frac{x}{2}-2\right)-2(x-30)=\tfrac{1}{7}(x-6)-7 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$$\left.\begin{aligned}(a^2-b^2)x-(a^2-ab+c^2)y&=a(a-2b)-\frac{bc^2}{a-b}\\ \frac{x}{a}+\frac{y}{b}&=\frac{2a}{a^2-b^2}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(3).$$

675. A, B, C, D are four railway stations; from B to C is $2\frac{2}{3}$ miles more and from C to D $5\frac{2}{7}$ miles less than from A to B. A train starts from A and travels at the rate of 14 miles an hour; at B an accident happens to the engine which causes a delay of 6 hours, and after this the train proceeds to C at half speed; there another delay of half-an-hour occurs, and then the train moves on to D at a speed further diminished by 1 mile. A man starts from A at the same time as the train and travels straight across country to D, a distance of 58 miles; including stoppages he averages 3 miles an hour and reaches D just with the train; what is the distance by rail from A to D?

---

676. Prove that

$$x^3+6(y+z)x^2+12(y+z)^2x+8(y+z)^3$$
$$=4(3x+2y+6z)y^2+(x+6y+2z)(x+2z)^2.$$

677. Simplify the fraction

$$\frac{8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3}{3(2a+b+c)(a+2b+c)(a+b+2c)}.$$

678. Find the square root of

$$a+8\sqrt{bc}+2b+4\sqrt{2ca}+8c+2\sqrt{2ab}.$$

679. Solve the equations:

$$\frac{(x-1)(x-2)(x-6)}{(x-3)^3}=1 \quad \ldots\ldots(1).$$

$$\left.\begin{aligned}\frac{3}{2x}+\frac{2}{3y}&=5\\ \frac{2}{x}+\frac{3}{y}&=13\end{aligned}\right\} \quad \ldots\ldots(2).$$

$$y+z-3x+a=z+x-3y+b=x+y-3z+c=0 \quad \ldots\ldots(3).$$

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680. If $a=4$, $b=3$, $c=2$, $d=1$, $e=0$; find the value of

$$3(b+d)\{6(a-d)^3+b(a-c)^2\}-(c+d)\{15(c-a)^3-d(a+c)^2\}$$
$$+(b+c)\{(b-3c)^3+(a-d)^3\}-(a+c)^2(b+c)^2de.$$

681. Find the H.C.F. of $3x^4-10x^3y+22x^2y^2-22xy^3+15y^4$ and $2x^4-7x^3y+16x^2y^2-17xy^3+12y^4$.

682. Simplify

$$\frac{a^4+b^4+ab(a^2+b^2)}{(a+b)^2}-\frac{a^4+b^4-ab(a^2+b^2)}{(a-b)^2}+\frac{12a^2b^2}{(a+b)^2-(a-b)^2}.$$

683. Extract the square root of

$$\frac{x^4}{y^4}+\frac{y^4}{x^4}-2\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)+3\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)-4\left(\frac{x}{y}+\frac{y}{x}\right)+5.$$

684. Solve the equations:

$$r(q+x)-pr=t(q+x)-pt \quad \ldots\ldots(1).$$

$$\frac{16x-27\frac{4}{5}}{3x-4}+\frac{77-x}{3(x-1)}=5+\frac{23}{x-1} \quad \ldots\ldots(2).$$

685. Two gangs of workmen are placing sleepers for parallel lines of railway; the first gang had placed 36 sleepers when the second began, and place eight sleepers to the second's seven, but have to place four sleepers in the same space in which the second place three. At what distance from the starting point will the one overtake the other, supposing there are 1764 of the more closely placed sleepers in a mile?

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686. If

$V=5a+4b-6c$, $X=-3a-9b+7c$, $Y=20a+7b-5c$, $Z=13a-5b+9c$;

calculate the value of $V-(X+Y-Z)$.

687. Divide a by $a-2x$ to six terms.

688. Separate into three factors the expression

$$12x^4+x^2y^2-y^4.$$

689. Reduce to their simplest forms:

$$\left(\frac{ap^2-aq^2+2bpq}{p^2-q^2}\right)^2+\left(\frac{bq^2-bp^2+2apq}{p^2-q^2}\right)^2 \quad \ldots\ldots\ldots\ldots(1).$$

$$\frac{2x^2-3y^2-z^2+4yz+zx+xy}{2x^2-12y^2+4z^2-8yz-9zx-5xy} \quad \ldots\ldots\ldots\ldots\ldots(2).$$

690. Solve the equations:

$$x-\frac{x-2}{2}=5\tfrac{3}{4}-\frac{x+10}{5}+\frac{x-2}{4} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

$$ax+by=1=bx-\frac{b}{a}+ay-\frac{a}{b} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$$\left.\begin{aligned} x+y+z&=1\\ ax+by+cz&=0\\ a^3x+b^3y+c^3z&=abc \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

691. A merchant engaged two writers, their pay being Rs. 60 for the first month, with a fixed monthly increase afterwards; they agreed to serve for one year, and each of them placed in the merchant's hands a deposit, to be forfeited in proportion to the part of the year during which he might not serve. One remained at his post $7\frac{1}{2}$ months and received for salary and portion of deposit returned Rs. 537, and the other remained 10 months and received in like manner Rs. 728. 5 as. 4 p.; what was the monthly increase, and what the amount of the deposit?

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692. If $a-b=0$, find the value of

$$(mb-nc)(mc-na)(ma-nb)+(nb-mc)(nc-ma)(na-mb).$$

693. Reduce to its simplest form $\dfrac{a^5+11a+12}{a^5+11a^3-54}$.

694. Shew that

$$\frac{b-c}{m+bc}+\frac{c-a}{m+ca}+\frac{a-b}{m+ab}=m\frac{(b-c)(c-a)(a-b)}{(m+bc)(m+ca)(m+ab)}.$$

695. Extract the square root of

$$(a+4)^4+8a^2(a-4)^2-256a^2.$$

696. Solve the equations:

$$\frac{3}{10x+9}+\frac{4}{45x+2}=\frac{7}{18x+5} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

$$\left.\begin{aligned} \frac{x}{a}+\frac{y}{b}&=2\\ ax-by&=a^2-b^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

697. A letter-carrier has to go daily from P to Q in a prescribed time; if he goes a mile an hour faster than his ordinary rate he arrives at Q half-an-hour before the time, but if he goes a mile an hour slower he arrives three-quarters of an hour late; find his ordinary rate and the distance from P to Q.

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698. Subtract $b\{a-(b+c)\}$ from the sum of $a\{a-(c-b)\}$ and $c\{a-(b-c)\}$.

699. Reduce to its lowest terms the fraction

$$\frac{x^4-4x+3}{2x^5-11x^2-9}.$$

700. Extract the square root of

$$x^4-2ax^3+5a^2x^2-4a^3x+4a^4.$$

701. Solve the equations:

$$\frac{x}{x-2}+\frac{9-x}{7-x}=\frac{x+1}{x-1}+\frac{8-x}{6-x} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\left.\begin{aligned}\frac{m}{x}+\frac{n}{y}&=a\\ \frac{n}{x}+\frac{m}{y}&=b\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

702. A set of bearers on a journey perform one-third of the distance at a certain rate and then halt an hour to take their food; the remainder of the journey is accomplished at only two-thirds of the former rate, and the bearers reach their destination in 7 hours after first starting. Had they travelled at the former rate $4\frac{1}{2}$ miles further than they did before halting, they might have halted $22\frac{1}{2}$ minutes longer and yet have reached the end of their journey in the same time; find the length of the journey.

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703. Simplify

$$24\{x-\tfrac{1}{2}(x-3)\}\{x-\tfrac{2}{3}(x+2)\}\{x-\tfrac{3}{4}(x-1\tfrac{1}{3})\};$$

and subtract the result from $(x+2)(x-3)(x+4)$.

704. Reduce $\dfrac{8xy-3x^2-4y^2}{7x^2y^2-2x^4+4y^4}$ to its simplest form.

705. Find the square root of

$$\frac{x^4}{4}-\frac{2x^3}{3}-\frac{11x^2}{36}+x+\frac{9}{16}.$$

706. Solve the equations:

$$\frac{(x-a)(x+b)}{x-a+b}=\frac{x(x-c)-b(x+c)}{x-b-c} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

$$\tfrac{1}{2}x+\tfrac{2}{3}y+\tfrac{1}{6}=\tfrac{1}{2}y+\tfrac{2}{3}x-7\tfrac{5}{2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

707. Two men A and B are employed on a piece of work which has to be finished in 14 days. In three days they do one-fifth of the work and then A's place is taken by C; B and C work for one day and do one-twentieth of the whole work and then B's place is taken by A, and A and C finish the work a day before the appointed time. Find the time in which the work could have been done (1) by each working separately, (2) by all working together.

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708. Prove that

$$(x-y)(x+1)(y+1)-x(y+1)^2+y(x+1)^2=(x-y)(x+y+2xy).$$

709. Find the H.C.F. of

$$x^4-6x^3+7x^2+6x-8 \text{ and } 2x^3-11x^2+11x+4.$$

710. Simplify

$$\frac{(x+y)^2+(x-y)^2}{(x+y)^2-(x-y)^2} \div \frac{x^4-y^4}{2xy(x-y)} \quad \ldots\ldots\ldots\ldots(1).$$

$$\frac{(a+1)^2}{(a-b)(a-c)}+\frac{(b+1)^2}{(b-c)(b-a)}+\frac{(c+1)^2}{(c-a)(c-b)} \quad \ldots\ldots\ldots\ldots(2).$$

711. Solve the equations:

$$\frac{1}{x-1}-\frac{1}{x}-\frac{1}{x+3}+\frac{1}{x+4}=0 \quad \ldots\ldots\ldots\ldots(1).$$

$$\left.\begin{aligned}\frac{b}{x}+\frac{a+c}{y}&=m\\ \frac{a-c}{x}+\frac{b}{y}&=n\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(2).$$

712. When a certain number of two digits is divided by the sum of its digits the quotient is 7, and the sum of the reciprocals of the digits is nine times the reciprocal of the product of the digits; what is the number?

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713. Divide

$$a^3+8b^3+27c^3-18abc \text{ by } a^2+4b^2+9c^2-6bc-3ca-2ab.$$

714. Resolve into three factors the expression

$$(x+1)(x+3)(x+5)(x+7)+15.$$

715. Prove the identity:

$$4(a^2+ab+b^2)^3-(a-b)^2(a+2b)^2(2a+b)^2=27a^2b^2(a+b)^2.$$

716. Find the H.C.F. and L.C.M. of

$$3x^4+17x^3+27x^2+7x-6 \text{ and } 6x^4+7x^3-27x^2+17x-3.$$

717. If $y=\dfrac{1+x}{1-x}$, prove that $\left(x-\dfrac{1}{x}\right)\left(y-\dfrac{1}{y}\right)=4\dfrac{xy+1}{x-y}$.

718. Solve the equations:

$$\frac{a+c}{x-2b}-\frac{b+c}{x-2a}=\frac{a-c}{x+2b}-\frac{b-c}{x+2a} \quad \text{.....(1).}$$

$$\left.\begin{aligned} 2x-3y+z+1&=0\\ 5x-3z&=6\\ 3x+2y&=4z \end{aligned}\right\} \quad \text{.....(2).}$$

719. In a quarter-of-a-mile race A gives B a start of 22 yards and beats him by 2 seconds, and in a 300-yards race he gives him a start of 2 seconds and beats him by $10\frac{1}{3}$ yards; find the rates of each.

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720. Simplify the expressions:

$$x-[a-\{2a-(3a-\overline{4a-x})\}] \quad \text{.....(1).}$$

$$\{(x^{a+b-c}\times x^{a-b+c})^{b}\}^{c} \quad \text{.....(2).}$$

$$(a+b)^m\times(a-b)^m\times(a^2+b^2)^m \quad \text{.....(3).}$$

721. Arrange the expression

$$x(p+x)\{p^2+q^2-x(p-x)\}-(p^2+qx)(2x^2-qx+q^2)$$

in descending powers of x, and divide it by $x^2+(p-q)x-p^2$.

722. Find the H.C.F. of

$$7x^3-19x^2+17x-5 \text{ and } 2x^4-x^3-9x^2+13x-5.$$

723. If $x^2=bc+ca+ab$, shew that

$$(a^2+x^2)(b^2+x^2)(c^2+x^2)$$

is a perfect square.

724. Simplify the expression

$$\frac{b^2+c^2-2a^2}{(a-b)(a-c)}+\frac{c^2+a^2-2b^2}{(b-c)(b-a)}+\frac{a^2+b^2-2c^2}{(c-a)(c-b)}.$$

725. Extract the square root of

$$1-xy-1\tfrac{5}{4}x^2y^2+2x^3y^3+4x^4y^4.$$

726. Solve the equations:

$$\frac{(x+1)(x+9)}{(x+2)(x+4)}=\frac{(x+6)(x+10)}{(x+5)(x+7)} \quad \text{.....(1).}$$

$$x+\sqrt{x^2+9}=9 \quad \text{.....(2).}$$

$$\left.\begin{aligned} 3y+4z&=6\\ 5z+6x&=21\\ x-2y&=5 \end{aligned}\right\} \quad \text{.....(3).}$$

727. A person bought 166 mangoes for ten rupees; some he bought at the rate of 18 per rupee and the rest at 15 per rupee; how many of each sort did he buy?

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728. Simplify the following expressions, arranging the last in ascending powers of x:

$$\{m-n-(3x-2y)\}-[3m+2n-\{x-y+(m+2n)-(2y-x)\}] \ldots (1).$$

$$\left(\frac{x^a}{x^b}\right)^{a+b} \div \left(\frac{x^{a+b}}{x^{a-b}}\right)^{\frac{a^2}{b}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$(x^2-xy+y^2)(x^2-2xy+y^2)(x^2+xy+y^2)(x^2+2xy+y^2) \ldots\ldots (3).$$

729. Find the H. C. F. of $6x^3+7x^2-9x+2$ and $8x^4+6x^3-15x^2+9x-2$, and write down the L. C. M. in factors.

730. Reduce to the simplest forms:

$$\frac{a^2-(b-c)^2}{(a-b)(a-c)}+\frac{b^2-(c-a)^2}{(b-c)(b-a)}+\frac{c^2-(a-b)^2}{(c-a)(c-b)} \ldots\ldots\ldots (1).$$

$$\frac{\frac{x^2+y^2}{y}-x}{\frac{1}{y}-\frac{1}{x}} \times \frac{x^2-y^2}{x^3+y^3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

731. Extract the square root of

$$x^2(x^2+y^2+z^2)+2x(y+z)(yz-x^2)+y^2z^2.$$

732. Solve the equations:

$$\frac{x-1}{x-2}+\frac{x-4}{x-5}=\frac{x-2}{x-3}+\frac{x-3}{x-4} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$\left.\begin{aligned} x-y+z&=1\\ x-2y+4z&=8\\ x-3y+9z&=27 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

733. A sum of Rs. 63. 4 as. was paid in rupees and two-anna pieces; the total number of coins being a hundred, how many of each kind were used?

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734. If a, b, c be three quantities whose sum is zero, prove that

$$a^4+b^4+c^4=2(b^2c^2+c^2a^2+a^2b^2).$$

735. Resolve into factors

$$b^3c+c^3a+a^3b-(bc^3+ca^3+ab^3).$$

736. Find the H. C. F. of

$$7x^4-2x^2-9x-2 \text{ and } 5x^3-6x^2-6x-11.$$

737. Simplify

$$\frac{(y-z)(y+z)^3+(z-x)(z+x)^3+(x-y)(x+y)^3}{(y+z)(y-z)^3+(z+x)(z-x)^3+(x+y)(x-y)^3}.$$

738. Extract the square root of

$$\frac{4x^2}{9a^2}+\frac{4x^2}{b^2}+\frac{9a^2x^2}{b^4}+\frac{9a^2}{4x^2}+\frac{9a^2}{b^2}+2.$$

739. Solve the simultaneous equations:

$$2x-\frac{y+3}{4}=7+\frac{3y-2x}{5},$$

$$4y+\frac{x-2}{3}=26\tfrac{1}{2}-\frac{2y+1}{2}.$$

740. In a half-mile race A gives B 22 yards' start and wins by 6 seconds; in a three-quarters of a mile race he gives B 20 seconds' start and is beaten by 29 yards 1 foot; in what time can each of them run a mile?

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741. Simplify the expressions:

$$7(a-3b+c)-[4(2b+4c)(6c-3b)-3(a-4b)(a+3b)+\{(5a-4b+3c)\times 4+a-47b+2c\}\div 7] \quad \ldots\ldots(1).$$

$$\frac{3x-12}{x^2-5x+6}+\frac{5x-3}{x^2-2x-3}-\frac{x+15}{x^2-5x-6} \quad \ldots\ldots(2).$$

$$\frac{\frac{a^3}{b^3}-\frac{b^3}{a^3}}{\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{a}{b}+\frac{b}{a}-1\right)}\div\frac{\frac{1}{a^2}+\frac{1}{ab}+\frac{1}{b^2}}{\frac{1}{b}-\frac{1}{a}} \quad \ldots\ldots(3).$$

742. Divide $\frac{1}{2}-x$ by $\frac{1}{4}-x+x^2$ to five terms.

743. Find the H.C.F. of

$x^4-8x^3+28x^2-53x+42$ and $x^4+6x^3-42x^2+129x-154$.

744. A composition of copper and tin containing 140 cubic inches weighs 42 lbs. 3 oz. avoirdupois; if a cubic inch of copper weigh $5\frac{1}{4}$ oz. and a cubic inch of tin $4\frac{1}{4}$ oz., how many ounces of each are in the composition?

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745. From $a(b+c)^2+b(c+a)^2+c(a+b)^2$ subtract

$(b+c)(a-b)(a-c)-(c+a)(b-c)(a-b)+(a+b)(a-c)(b-c)$.

746. Divide

$x^4+5ax^3+(25a-b-29)x^2-5(4a+b-4)x+4b$ by x^2+5x-4;

and shew that $(4x^2-5x+7)^2-(5x^2+14x+2)^2$

is divisible by x^2+x+1, giving the quotient.

747. Resolve into four factors

$$(1+y)^2-2(1+y^2)x^2+(1-y)^2x^4.$$

748. Find the H.C.F. of

$6x^5-4x^4-11x^3-3x^2-3x-1$ and $4x^4+2x^3-18x^2+3x-5$.

749. Extract the square root of

$x^{4m+2}+6x^{3m+1}-10x^{2m+1}y^{m-2}+9x^{2m}-30x^my^{m-2}+25y^{2m-4}$.

750. Solve the equations:

$$\frac{3x-2}{5}+\frac{4x-1}{7}-\frac{10x}{9}=5(x-9)+3-\frac{x}{3} \quad \text{.....(1).}$$

$$\left.\begin{aligned}49x-57y&=172\\57x-49y&=252\end{aligned}\right\} \quad \text{.....(2).}$$

$$\left.\begin{aligned}\frac{x}{a}+\frac{y}{b-a}&=5m\\\frac{x}{b}+\frac{y}{a-b}&=7m\end{aligned}\right\} \quad \text{.....(3).}$$

$$\left.\begin{aligned}\frac{x}{a}+\frac{y}{b}&=1-\frac{x}{c}\\\frac{y}{a}+\frac{x}{b}&=1+\frac{y}{c}\end{aligned}\right\} \quad \text{.....(4).}$$

Chap. XIII.—XV.

Solve the quadratic equations:

751. $3-x=\frac{2}{x}$.

752. $(x-a)(x-b)=ab-x^2$.

753. $6x+\frac{35-3x}{x}=44$.

754. $\frac{1}{a+b+x}=\frac{1}{a}+\frac{1}{b}+\frac{1}{x}$.

755. $\frac{a}{x+a}+\frac{b}{x+b}=\frac{a-c}{x+a-c}+\frac{b+c}{x+b+c}$.

756. $\frac{90}{x}=\frac{90}{x+1}+\frac{27}{x+2}$.

757. $\frac{2x-1}{2x+1}+\frac{2x+1}{2x-1}=3$.

758. $\frac{x-3}{x-4}+\frac{x-4}{x-3}=2\frac{1}{20}$.

759. $\frac{x}{12}+\frac{12}{x}=\frac{5}{8}(x-2)$.

760. $x+\frac{24}{x-1}=3x-4$.

761. $\frac{1}{x-1}-\frac{4}{x-2}+\frac{4}{x-3}-\frac{1}{x-4}=\frac{1}{30}$.

762. $x^2+\frac{1}{x^2}+x+\frac{1}{x}=4$.

763. $\frac{3}{5-x}+\frac{2}{4-x}=\frac{8}{2+x}$.

764. $x^2-x+113=(3+x)(31-4x)$.

765. $x^{n-1}+x^{n+1}=2x^n$.

766. $nax+mbx=mnx^2+ab$.

767. $x^2-a^2+\frac{1}{x^2}-\frac{1}{a^2}=0$.

768. $x^2+3x-10=ax^2+bx^2+cx^2+5(a+b+c)x.$

769. $\sqrt{x+4}+\sqrt{2x+9}=\sqrt{3x+25}.$ 770. $x+\sqrt{5x+10}=8.$

771. $4x^2-6x+3\sqrt{2x^2-3x+7}=44.$

772. $x^2+7x+4=5\sqrt{x^2+7x+28}.$

773. $\sqrt{x^2+4x+4}=3x-2.$ 774. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\sqrt{m}.$

775. $\sqrt{a+b-x}+\sqrt{3a-b-3x}=2\sqrt{2b}.$

Solve the simultaneous equations:

776. $x^2-xy+y^2=21,\ x+y=6.$ 777. $x^2+xy+y^2=28,\ x-y=2.$

778. $x-\dfrac{x-y}{2}=4,\quad y-\dfrac{x+3y}{x+2}=1.$

779. $3x^2-4xy+y^2=20,\quad 2x^2-7y^2=4.$

780. $x^2+xy+4y^2=6,\quad 3x^2+8y^2=14.$

781. $x+2y=5,\quad x^2+2y^2=11.$

782. $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{2},\quad xy=1.$ 783. $x^2+y^2=a^2,\quad xy=b^2.$

784. $(a+b)x-(a-b)y=\dfrac{3a+7b}{21}(x^2-y^2),$

$(a-b)x+(a+b)y=\dfrac{7a-3b}{21}(x^2-y^2).$

785. $x(y+z)=22,\quad y(z+x)=40,\quad z(x+y)=42.$

786. If α, β be the roots of $x^2+px+q=0$, prove that the roots of $4x^2+2px+q=0$ are $\dfrac{\alpha}{2}, \dfrac{\beta}{2}$.

787. If x_1, x_2 be the roots of $ax^2+bx+c=0$, shew that

$$\frac{x_1}{x_2}+\frac{x_2}{x_1}=\frac{b^2-2ac}{ac}.$$

788. If α be one root of the equation $4x^2+2x-1=0$, prove that $4\alpha^3-3\alpha$ is the other root.

789. If x_1, x_2 be the roots of the equation

$$x^2+mx+m^2+n^2=0,$$

prove that $x_1^4+x_1^2x_2^2+x_2^4=n^2(2m^2+3n^2).$

790. If α, β be the roots of the equation

$$a(m^2+x^2)+amx+cm^2x^2=0,$$

shew that $a(\alpha^2+\beta^2)+a\alpha\beta+c\alpha^2\beta^2=0.$

If $a : b :: c : d$; prove that

791. $4(a+b)(c+d)=bd\left(\frac{a+b}{b}+\frac{c+d}{d}\right)^2$.

792. $ad^2(ma+nb)=cb^2(mc+nd)$.

793. $\frac{a^2+c^2}{b^2+d^2}=\frac{ac}{bd}$.

794. $\frac{a^2+b^2}{a^2-b^2}=\frac{ac+bd}{ac-bd}$.

795. $\frac{ma+nc}{mb+nd}=\sqrt{\frac{a^2+c^2}{b^2+d^2}}$.

796. $\frac{a+c}{b+d}=\sqrt[3]{\frac{a^3+c^3}{b^3+d^3}}$.

797. $\frac{(a+c)^3}{(b+d)^3}=\frac{a(a-c)^2}{b(b-d)^2}$.

798. $\frac{a^2\pm c^2}{b^2\pm d^2}=\left(\frac{a\pm c}{b\pm d}\right)^2$.

799. If $a : b :: b : c :: c : d$, prove that
$$(ab+bc+cd)^2=(a^2+b^2+c^2)(b^2+c^2+d^2).$$

800. If $\frac{b-c}{a^2}=\frac{c-a}{b^2}=\frac{a-b}{c^2}$, shew that a, b, c are equal to each other.

801. Given that
$$\frac{b+c-a}{b+c}=\frac{c+a-b}{c+a}=\frac{a+b-c}{a+b},$$
and that the sum of a, b, c is not zero, prove that
$$a=b=c.$$

802. A and B compared their monthly incomes and found that A's income was to that of B as 7 to 9, and that the third part of A's income was greater by Rs. 30 than the difference of their incomes; what was the income of each?

803. A certain number consists of two digits; the left-hand digit is double the right-hand digit, and if the digits be reversed the ratio of the number thus formed to 60 is 4 : 5; find the number.

804. Two armies number 11000 and 7000 men respectively, and before they fight each is reinforced by 1000 men; in favour of which army is the increase?

805. Find two numbers the greater of which shall bear to the less the same ratio that their sum bears to 42 and their difference to 6.

806. A and B speculate with different sums; A gains Rs. 150, B loses Rs. 50, and A's stock is now to B's as 3 to 2; but if A had lost Rs. 50 and B had gained Rs. 100, then A's stock would have been to B's as 5 to 9. What was the stock of each?

807. The sum of two numbers is 100, and the greater is to the less as 7 to 3; what are the numbers?

808. A number consists of two digits whose sum is 8, and a second number is obtained by reversing the digits; if the product of the two numbers is 1855, find the original number.

809. A man selling a horse for £72 finds that his loss per cent. is one-eighth of the number of pounds he gave for the horse; what was the cost price?

810. A merchant bought a number of pieces of cloth for £32 and sold them at £4. 10*s.* each, thus gaining as much as one piece cost him; how many pieces were there?

811. Shew that it is impossible to divide 10 into two parts such that their product may be 30.

812. A and B start simultaneously from opposite ends of a straight course, each walking uniformly and A (who is the faster walker) at the rate of four miles an hour, and they meet at the end of two hours. If, when A reached the middle point of the course, they had interchanged their rates of walking, they would have met a quarter of a mile nearer the middle point; find B's rate of walking and the length of the course.

813. Find a number of two digits such that if it be divided by the product of the digits the quotient is 2, and if 27 be added to the number the order of the digits is reversed.

814. The fore-wheel of a carriage makes six revolutions more than the hind-wheel in going 120 yards; but, if the circumference of each were increased by a yard, it would make only four revolutions more than the hind-wheel in the same distance; find the circumference of each wheel.

815. Divide 32 into two such parts that if 6 be divided by each part the sum of the quotients may be unity.

816. Two railway trains start at the same hour from two stations and proceed each towards the other station; when they meet it is found that one of them has travelled 84 miles more than the other, and that, by continuing to travel each at the same rate as it had done before, the one will complete the journey in 9 hours and the other in 16 hours; find the distance between the stations and the rates at which the trains travelled.

817. The length of a certain rectangular field is to its breadth as 6 : 5, a sixth part of the field is planted and 625 square yards remain for ploughing; what are the dimensions of the field?

818. Two messengers A and B were despatched to the same place 90 miles distant; A started an hour after B but by riding a mile an hour faster than B arrived at his destination at the same time as B; at what rate did A travel?

819. A farmer bought 5 oxen and 12 sheep for £63, and for £90 could have bought four more oxen than he could have bought sheep for £9; what did he pay for each?

820. A vessel can be filled by two pipes both open at once in $4\frac{4}{5}$ hours, and by one of the pipes alone the vessel would be filled 4 hours sooner than by the other; find the time which each pipe alone would take to fill the vessel.

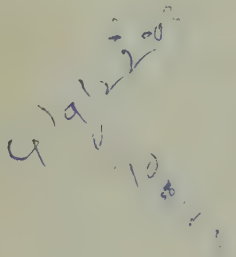
821. Two labourers A and B were employed at different rates of wages, paid at the end of a certain time; A received Rs. 16. 14 as., and B, who had worked 15 days less than A, received Rs. 7. 8 as. If B had worked all the time and A had omitted 15 days, they would each have received the same sum; how many days did each work, and what were the wages of each?

822. Some bees were clustered on a lotus; at one time the square root of half their number flew off to a jasmine; afterwards eight-ninths of the whole flew away, and two bees were left on the lotus. How many were there at first?

823. Divide 250 into two parts such that the sum of their square roots shall be 20.

824. A tailor bought some cloth for Rs. 120; if he had bought three yards less for the same money the cloth would have cost two rupees a yard more than was actually paid for it; how many yards were bought?

825. A boy bought a number of oranges for two rupees, but if he had bought eight more for the same money he would have paid four pies less for each; how many did he buy?



ANSWERS.

CHAPTER I.

1. 10. 2. 3. 3. 10. 4. 16. 5. 3. 6. 125. 7. 2. 8. 140.
9. 100. 10. 16. 11. 48. 12. 1125. 13. 640. 14. 3.
15. 160. 16. 1600. 17. 192. 18. 6250. 19. 128. 20. 1.
21. 0. 22. 0. 23. 0. 24. 236. 25. 1. 26. 138. 27. 24.
28. 22. 29. 593. 30. 159. 31. 14. 32. 2. 33. 39.
34. 17. 35. 0. 36. 0. 37. 0. 38. 24. 39. 6. 40. 22.
41. 10. 42. 3. 43. 46. 44. 2. 45. 0. 46. 2. 47. 4.
48. 4. 49. $\frac{3}{4}$. 50. 1. 51. 3. 52. 6. 53. 9. 54. 1.
55. 3. 56. 1. 57. 0. 58. 0. 59. 0. 60. 0. 61. 3.
62. 0. 63. 4. 64. 0. 65. 1. 66. 6. 67. 1. 68. 2.
69. 1. 70. 0. 71. 0. 72. 0. 73. 6. 74. 1. 75. 2.
76. 1. 77. 3. 78. 1. 79. 1. 80. 6. 81. 5. 82. 3.
83. 0. 84. 1. 85. 0. 86. 0. 87. 24. 88. 0. 89. 0.
90. 0. 91. 0. 92. 2. 93. 0. 94. 2. 95. 1.
98. 1st and 4th, 2nd and 6th. 99. 5, 6, 5, 6, 6, 7. 100. a^3, b^3, c^3, a^2b, a^2c, ab^2, b^2c, ac^2, bc^2, abc.

CHAPTER II.

1. $2a$. 2. 0. 3. $a+b$. 4. a. 5. $a+b+c$. 6. b.
7. $10a+b-c-3d$. 8. $a+b+c$. 9. $a-2b+3c-4d$. 10. a^2.
11. 0. 12. $a^3+3a^2b-2ab^2-4b^3$. 13. 0. 14. a^4.
15. $a^4+b^4+c^4$. 16. $2x^2-6x+1$. 17. x^2+x. 18. $3x^2+2y^2$.
19. $2y^2$. 20. $2y^2+2y-2$. 21. x^2+x+1. 22. 0.
23. $4ab-7ax^2+a^2x$. 24. 1. 25. $x^4+x^3+x^2+x$. 26. x^4.
27. x^4+x^3-x-1. 28. $x^5-5ax^4+10a^2x^3-10a^3x^2+5a^4x-a^5$.
29. y^3-y^2+y-1. 30. $9a^3+a^2-4a+7$. 31. $4a^3+4a^2+4a+4$.
32. $x^2+y^2+z^2$. 33. $5x^3-7x^2y+xy^2+4y^3$. 34. $x^2-2xy+3y^2-4x+5y-6$.
35. $2x^4+2x^3y-2x^2y^2-2xy^3+2y^4$. 36. $x^3+y^3+z^3$. 37. $\frac{1}{2}a-\frac{1}{10}b$.

38. 0. 39. $-\frac{1}{6}x^2-\frac{1}{12}x-\frac{1}{8}$. 40. x^2-y^2. 41. $\frac{1}{6}yz-\frac{1}{7}zx-\frac{1}{6}xy$.
42. 0. 46. $-$Rs. 200, $-$Rs. 300, 0. 47. $-$£$(x-yz)$.
48. At the starting point. 49. -6 miles. 50. -5 minutes P.M.

CHAPTER III.

1. $2b$. 2. $a+b$. 3. $2b+2c$. 4. $-a+2b-5c$.
5. $-3a-3b+3c$. 6. $2a-2b-2c$. 7. $a+\frac{1}{10}b-\frac{1}{4}c$. 8. $-x+y$.
9. $2x-2y$. 10. $8y+3z$. 11. $4x-5y-2z$. 12. $3x+3y+3z$.
13. $-4x+4y+6z$. 14. $\frac{1}{6}x-\frac{1}{15}y+\frac{1}{2}z$. 15. $\frac{3}{2}x-\frac{1}{4}y-z$.
16. $-2x^2-2x+2$. 17. x^2+x-1. 18. $x^2+ax+a^2-b^2$.
19. $ax^2-bx^2-ax+a^2$. 20. $9x^2+5xy+y^2$. 21. $2a^2+2a+2$.
22. $4-4a+4a^2+4a^3+4a^4$. 23. $2x^3-2x^2+2x-1$. 24. x^3+x^2-2x.
25. $x^4+ax^3+a^2x^2-a^3x-a^4$. 26. $2yz+2xy$. 27. $2bc-2ca+2ab$.
28. $4a^3-4a^2x+4ax^2-4x^3$. 29. $\frac{7}{12}a^2-\frac{11}{15}ab+\frac{7}{8}b^2$. 30. $\frac{11}{15}qr-\frac{3}{4}rp+\frac{7}{18}pq$.
31. 1. 32. $3a-3b$. 33. 0. 34. 0. 35. 0. 36. 0.
37. 1. 38. x^3-x^2+x-1. 39. $x+y$. 40. $x^2-y^2-z^2$.
41. -2. 42. 2. 43. 2. 44. 1. 45. 0. 46. $2a^2+b^2-4c^2$.
47. $x^4-4x^3y+3x^2y^2-3xy^3+y^4$. 48. $2a^2-3ab+2b^2$.
49. $2x^3-7x^2y+7xy^2-2y^3$. 50. $z-(x-y)=2$.

CHAPTER IV.

1. $14a^2$. 2. $-20a^4$. 3. $-\frac{3}{4}a^4$. 4. $-32x^4$. 5. $18x^5$.
6. $-\frac{1}{6}x^3$. 7. $x^2-7x+10$. 8. x^2+5x-6. 9. $x^2+6x-40$.
10. x^3-x^2-4x+4. 11. x^3-64. 12. $x^3-11x+12$. 13. $3xy-6y^2$.
14. $\frac{1}{4}xy-\frac{3}{16}y^2$. 15. $5x^2-\frac{21}{4}xy+y^2$. 16. $13ab-6a^2-6b^2$.
17. $8a^3-20a^2b-14ab^2+35b^3$. 18. $5a^4-11a^3b+2a^2b^2$.
19. $23xy-6x^2-20y^2$. 20. $\frac{4}{3}x^2-\frac{17}{6}xy+\frac{5}{8}y^2$. 21. $10x^3-5x^2y-6xy^2+3y^3$.
22. $-3a^3+19a^2b-13ab^2-35b^3$. 23. $-4a^3-5a^2b+18ab^2-9b^3$.
24. $4x^3-14x^2y+10xy^2+4y^3$. 25. $3x^3-13x^2y+14xy^2-6y^3$.
26. a^4-1. 27. $8a^4-18a^3+19a^2-19a+6$. 28. $6a^4-10a^3+7a^2-5a+2$.
29. $x^5-8x^4+16x^3-24x^2+71x+120$. 30. $2x^3-x^2-11xy^2+4xy-4y^2+6y^3$.
31. $x^3-2x^2(y-1)-x(3y-2)+y^3-1$. 32. $2a^2x^3-5a^3x^2-8a^4x+5a^5$.
33. $4a^4-7a^2bx+(5a^3b-4a^2b^2+3b^2)x^2-(5ab^2-3b^3)x^3-5ab^3x^4$. 34. x^4-a^4.
35. $x^3+y^3+z^3-3xyz$. 36. $2a^4-5a^3b-2a^2b^2+11ab^3-6b^4$.
37. $-2x^4+3x^3y-4x^2y^2+xy^3+2y^4$. 38. $x^4+2bx^3-2(a^2-b^2)x^2$
$+(a^3-a^2b+ab^2+b^3)x-a^2b^2$. 39. $x^5-ax^4+6a^3x^2-2a^4x-a^5$.
40. $x^6+x^5y-xy^5-y^6$. 41. $49x^2-64y^2$. 42. $(x+z)^2-y^2=x^2+2zx-y^2+z^2$.
43. $16x^2-(3y-2z)^2=16x^2-9y^2-4z^2+12yz$. 44. x^3-1. 45. x^3+1.
46. $8a^3+27b^3$. 47. $125a^3-8b^3$. 48. $(a^2+a)^2-(b^2-b)^2=a^4+2a^3+a^2$
$-b^4+2b^3-b^2$. 49. $(a^2-c^2)^2-(b^2+2bc)^2=a^4-b^4+c^4-4b^3c-4b^2c^2-2a^2c^2$.

50. $4a^2b^2-(a^2-b^2+c^2)^2=2b^2c^2-2c^2a^2+6a^2b^2-a^4-b^4-c^4$.

51. $(a^3+a^2b)^2-(ab^2-b^3)^2=a^6+2a^5b+a^4b^2-a^2b^4+2ab^5-b^6$.

52. $4x^4+1$. 53. $x^4-(3xy-2y^2)^2=x^4-9x^2y^2+12xy^3-4y^4$.

54. $(x^2+y^2)^2-5x^2y^2=x^4-3x^2y^2+y^4$. 55. $3x^2y^2-(x^2+y^2)^2=x^2y^2-x^4-y^4$.

56. $(7x-3y)^2\,(7x+3y)^2=(49x^2-9y^2)^2$ &c.

57. $(x^3+x)^2-(x^2+1)^2=x^6+x^4-x^2-1$.

58. $(a^3+2ab^2)^2-(4a^2b+b^3)^2=a^6-12a^4b^2-4a^2b^4-b^6$.

59. $a\,(a^2-4b^2)-b\,(a+2b)^2=a^3-a^2b-8ab^2-4b^3$. 60. $a^8+a^4b^4+b^8$.

61. $4b^2c^2-(b^2+c^2-a^2)^2=2\,(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4)$.

62. a^8-b^8. 63. x^8+x^4+1. 64. $(x^n-1)^2\,(x^n+1)^2\,(x^{2n}+1)^2=(x^{4n}-1)^2$ &c.

65. x^6-y^6. 66. $9a^2-24ab+16b^2$. 67. $25x^4+70x^2y^3+49y^6$.

68. $a^2+b^2+c^2-2bc+2ca-2ab$. 69. $27a^3+54a^2b+36ab^2+8b^3$.

70. $x^3-15x^2y+75xy^2-125y^3$. 72. $as+bc=(a+b)\,(c+a)$ &c.

78, 79. Let $s-a=x$, $s-b=y$, $s-c=z$; then $x+y+z=0$, $\therefore -x=y+z$ &c.

80. Each expression $=abc$. 83. Let $a=y-z$, $b=z-x$, $c=x-y$.

84. Each expression $=bcd+acd+abd+abc$.

85. Given expression $=\frac{1}{2}\{(y-z)^2+(z-x)^2+(x-y)^2\}$.

91. 1; given expression $=11x\,(x^2+x+1)-3\,(x^2+x+1)+1$.

92. 0; given expression $=9x\,(x^2-x+1)+5\,(x^2-x+1)$.

93. 0; in this and the two following examples employ the artifice of Ex. 91, 92. 94. 1. 95. 1.

96. 1; given fraction $=\dfrac{x\,(3x+1)^2-3x}{2x\,(3x+1)+1}$. 97. 1. 98. 1.

99. 2. 100. 1.

CHAPTER V.

1. $2a^2$. 2. $-4a^2b^2$. 3. $4a$. 4. $-5x^2$. 5. $6x^3y$. 6. $7a-3$.

7. $4a-b$. 8. $-2x+1$. 9. $\frac{5}{4}x-\frac{3}{2}$. 10. $2x-3y$. 11. $-z+y+x$.

12. $-a^2+a+1$. 13. $-2a^2+6ab-9b^2$. 14. $4x^2-3x+2$.

15. $-3x^2+9x+5$. 16. $a+2$. 17. $a+6$. 18. $2-3a$. 19. $a+4b$.

20. $x-7$. 21. $x-11$. 22. $x-3$. 23. $-4x+3$. 24. $2x-3$.

25. $4x+3$. 26. $8x+3$. 27. $7x+12y$. 28. $-3x+4y$. 29. $\frac{5}{6}x+\frac{1}{2}$.

30. $\frac{7}{6}x+\frac{1}{2}y$. 31. $x+5$. 32. a^2-5a+6. 33. $-2a+b$.

34. $x^2-8x+15$. 35. $4x+3y$. 36. $5x^2+5xy-3y^2$.

37. $125x^3+25x^2y+5xy^2+y^3$. 38. $x-1$. 39. $x^4+x^3y+x^2y^2+xy^3+y^4$.

40. $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5$. 41. $x^5+2x^4+4x^3+3x^2+6x+12$.

42. $a+3$. 43. $2a-b$. 44. $x+1$. 45. $2x^2+3xy+y^2$.

46. $8a^2+11ab-b^2$. 47. $x+2y$. 48. $x-2$. 49. $x^2-(3b+2c)\,x+6bc$.

50. $ax^2+bxy+cy^2$. 51. $5a-3b$. 52. $7a-2b$. 53. $4x+9y$.

54. $6x+10y$. 55. $\frac{1}{2}x-\frac{2}{3}y$. 56. $x^2y^2+axyz+a^2z^2$. 57. $4a^2b^2-2ab+1$.

58. $3a-4b$. 59. $5x+7y$. 60. $x+y-z$. 61. $2a+3b+5c$.

62. a^2-a+1. 63. x^2+2x+3; $x^4+2x^2+9=(x^2+3)^2-4x^2$.
64. $(x+2)^2$; $x^4-8x^2+16=(x^2-4)^2$ and $x^2-4x+4=(x-2)^2$.
65. $x^2+2xy+7y^2$; $x^4+10x^2y^2+49y^4=(x^2+7y^2)^2-4x^2y^2$.
66. $2x^2-2x+1$; $4x^4+1=(2x^2+1)^2-4x^2$. 67. x^2-y^2. 68. $x+y-1$.
69. $2x^2+xy+y^2$; $4x^4+3x^2y^2+y^4=(2x^2+y^2)^2-x^2y^2$. 70. x^m-y^m.
71. x^m+y^m. 72. $(y-z)^2-(y-z)(z-x)+(z-x)^2=$&c.
73. $2x+3y-z$; $4x^2-9y^2-z^2+6yz=4x^2-(3y-z)^2$.
74. $b+c$. 75. $(a+b+c)^2+a(a+b+c)+a^2=$&c.
76. $(2b-c+2a)^2-(2b-c+2a)(2c-b-a)+(2c-b-a)^2=$&c.
77. $a^2-b^2-c^2+2bc$; $a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2=(a^2-b^2-c^2)^2-4b^2c^2$.
78. $(x-y)(x+y)^2$. 79. $(x^2-xy+y^2)(x^4-x^2y^2+y^4)$.
80. $3x^2+xy-y^2$; $9x^4-7x^2y^2+y^4=(3x^2-y^2)^2-x^2y^2$.
81. $x^2-x(y+z)+(y+z)^2$. 82. $(x-y)^2+(x-y)z+z^2$. 83. $x^2+2xy+2y^2$.
84. $(x^2+xy+y^2)^2$; $x^6-2x^3y^3+y^6=(x^3-y^3)^2$.
85. $yz-zx+xy$; $y^2z^2+z^2x^2-x^2y^2-2xyz^2=(yz-zx)^2-x^2y^2$.
92. No. 93. No. 94. Yes. 95. No. 96. No. 97. 5.
98. 11. 99. 17. 100. -18.

CHAPTER VI.

1. $(a-b)(a+b+1)$. 2. $(a+b)(c+d)$. 3. $(a-b)(c-d)$.
4. $(a-b)^2-(b-c)^2=(a-c)(a-2b+c)$. 5. $(b-c)(a-b-c)$.
6. $(a+b+c)^2-c^2=(a+b)(a+b+2c)$. 7. $(a+b)(a-b)^2$.
8. $(a-b)(a^2+b^2)$. 9. $(b-c)(bc-ca-ab)$.
10. $(a-b)^3-1=(a-b-1)\{(a-b)^2+a-b+1\}$. 11. $(a-b)(a^2-ab+b^2)$.
12. $(b-c)(a-b-c-x)$. 13. $(a+b)(a^2+mab+b^2)$.
14. $(a-b)(a+b)(ma+nb)$.
15. $(a+b)^2-(c+d)^2=(a+b+c+d)(a+b-c-d)$.
16. $(a^2-b^2-c^2)^2-4b^2c^2=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.
17. $(x-2)(x-7)$. 18. $(x+4)(x-9)$. 19. $(x+7)(x+9)$.
20. $(x-4)(x-6)$. 21. $(x+12y)(x-13y)$. 22. $(x-8a)(x+11a)$.
23. $(x+a+b)(x+a-b)$. 24. $(x-a-b)(x+a-b)$. 25. $(x+2)(4x-7)$.
26. $(x+5)(3x-8)$. 27. $(x-6)(2x+9)$. 28. $(3x-4y)(4x+5y)$.
29. $(ax-b)(bx-a)$. 30. $b(a^2x^2-1)-(ax-1)=(ax-1)(abx+b-1)$.
31. $c^2x^2-b^2-a(cx-b)=(cx-b)(cx+b-a)$.
32. $(mx-ny)^2-x^2=\{(m-1)x-ny\}\{(m+1)x-ny\}$.
33. $(x^2+5yz)(x^2-7yz)$.
34. $x(x^2-1)-12(x-1)=(x-1)(x-3)(x+4)$.
35. $x^4-(x-1)^2=(x^2-x+1)(x^2+x-1)$.
36. $x^2(x-2)+x(x-2)-6(x-2)=(x-2)^2(x+3)$.
37. $(x^2-1)^2-16x^2=(x^2-4x-1)(x^2+4x-1)$.
38. $(x^2+2)^2-x^2=(x^2-x+2)(x^2+x+2)$.

39. $(x^2+3y^2)^2-4x^2y^2=(x^2-2xy+3y^2)\,(x^2+2xy+3y^2)$.
40. $bx^2\,(ax-1)+a^2x^2-1=(ax-1)\,(bx^2+ax+1)$. 49. Put $x=0$.
50. $(y-z)\,(z-x)\,(x-y)$; $b-c=y-z$, $c-a=z-x$, $a-b=x-y$.

CHAPTER VII.

1. a^2b^2. 2. $3a^3b^2c$. 3. $2ab^2c^3$. 4. $5x^2y$. 5. $13x^2y^2z$.
6. $3x^2y^2z^2$. 7. $2a$. 8. $19abc$. 9. $a^2b^2c^2$. 10. $8y^2z$.
11. $a+b$. 12. $3\,(a+b)$. 13. $2a\,(a^2+a+1)$. 14. a^2+ab+b^2.
15. $8ab^2\,(a^2+b^2)$. 16. $a+b$. 17. $a-b$. 18. $c-d$.
19. $a+b+c$; given expressions are $(a-b)\,(a+b+c)$, $a^2-(b+c)^2$.
20. $a+b-c$; given expressions are $(a+b)^2-c^2$, $a^2-(b-c)^2$.
21. $2a+b+c$; given expressions are $(a+b+c)^2-a^2$, $a\,(2a+b+c)$.
22. a^2+b^2. 23. $a-b$.
24. a^2+ab+b^2; $(a+b)^4-a^2b^2=(a^2+2ab+b^2)^2-a^2b^2$. 25. $a^2+b^2-c^2$.
26. $x-3$. 27. $x+1$. 28. $x+12$. 29. $x-5$. 30. $x+y$.
31. $x-15a$. 32. $2\,(x+7)$. 33. $3x-2$. 34. $7x+3$. 35. $x+7$.
36. $3x-7$. 37. $7x-2$. 38. $2x-3y$. 39. $3x-y$. 40. $x-6a$.
41. $ax-8y$. 42. $3x\,(2x+1)$.
43. $x-1$; $x^3-4x^2+3=x^2\,(x-1)-3\,(x^2-1)$ &c.
44. $x-1$; $x^3-4x+3=x^3-1-4\,(x-1)$ &c. 45. x^2-2.
46. x^2-8. 47. x^2-3x+1; $x^4-7x^2+1=(x^2+1)^2-9x^2$ &c.
48. $x-3$. 49. $x-5$. 50. $x-2$. 51. $ax-2b$. 52. $x-3a$.
53. $x-1$. 54. x^2-xy+y^2. 55. $3x^2y^2\,(x-y)$. 56. $x-a$.
57. $x+a$. 58. $(x+y+z)\,(x+y-z)$. 59. $xy\,(x^2+xy+y^2)$.
60. x^2-y^2. 61. $a+b$. 62. $a-2b$. 63. $a-1$. 64. a^2+b^2.
65. $x+3$. 66. $(x+2)^2$. 67. $(x-2)\,(x+5)$. 68. x^2+3x+1.
69. x^2-x+1. 70. x^2-4. 71. $x+a$. 72. $2x-3$. 73. $x+1$.
74. $x-2$. 75. $x+y$. 76. $x-3$. 77. $3x\,(x-1)$. 78. $x-2$.
79. $(x-y)^3$. 80. $(x-y)\,(3x+y)$. 81. $x-2$.
82. $(x+y)\,(x^2-2xy+3y^2)$. 83. $(x-1)^2\,(x+2)$. 84. x^2+ax+a^2.
85. $2x^2\,(x+1)\,(x-2)$. 86. $x-3$; subtract. 87. $x-9$; subtract.
88. $2x-1$; multiply first expression by 3 and subtract.
89. $4x+1$; multiply first expression by 2 and subtract.
90. $x-3$; multiply second expression by 2 and subtract.
91. $x-2$; add and subtract.
92. $x-1$; subtract, and multiply first expression by 2 and subtract.
93. $x-2$; add. 94. Add and subtract.
95. Subtract, and multiply second expression by 2 and subtract.
96. Multiply first expression by 5, second by 2 and add; multiply first by 2, second by 5 and subtract.
97. 2. 98. 3. 99. 0 or 6. 100. 0 or 9.

CHAPTER VIII.

1. a^4b^6. 2. $12a^3b^4c^2$. 3. $42x^2y^3$. 4. $30x^3y^4$. 5. $18a^4b^3c^5$.
6. $500a^4b^4c^4$. 7. $170a^4x^4y^4$. 8. $12x^3y^4z^2$. 9. $72x^5y^5z^5$.
10. $54x^6y^4z^4$. 11. $(a-b)(a^3+b^3)$. 12. $12(a-b)(a^3-b^3)$.
13. $(a+b)(a^3-b^3)$. 14. $6a^2(a^3+1)$. 15. $a(a^3+b^3)$. 16. a^6-b^6.
17. a^6-b^6. 18. $ab(a+b)^3$. 19. $c(b-a)(a+b-c)$.
20. $c(a-b)(a+b-c)(bc+ca-ab)$. 21. $(a-b)(c-d)(c+d)$.
22. $(a+b+c)(a-b-c)(a+b-c)$. 23. $(a+b)(a-b)(a+b+2c)$; given expressions are $(a+b+c)^2-c^2$, $(a+c)^2-(b+c)^2$. 24. $a(a-b)^3$.
25. $60ab(a^4-b^4)(a^4-a^2b^2+b^4)$. 26. $(x-1)(x-2)(x+3)$.
27. $(x-1)(x+5)(x-9)$. 28. $(x-3)^2(x+8)$.
29. $(x+2y)(x+3y)(x-4y)$. 30. $(x-7a)(x+11a)(x-15a)$.
31. $(x-3a)(x+11a)(x+12a)$. 32. $6x(x-2)(x-3)(x+9)$.
33. $2(x-2)(2x-1)(3x+2)$. 34. $(x-2)(3x-7)(4x+9)$.
35. $(x+8)(x+12)(2x-1)$. 36. $(2x+1)(3x+2)(4x-3)$.
37. $(2x-5y)(3x-8y)(3x+8y)$. 38. $3(x-y)(2x-3y)(3x+4y)$.
39. $(2x+a)(3x+a)(4x-5a)$. 40. $(x-a)(x+a)(11x-12a)$.
41. $(2a+7)(5a-3)(6a-1)$. 42. $(a+2b)(9a-13b)(10a-11b)$.
43. $(ax+b)(bx-a)(bx+a)$. 44. $(x-a)(x+a)(ax-b^2)$.
45. $12x(ax+3)(4ax-1)(3ax-1)$. 46. a^6-b^6.
47. $(x-1)(x+1)(x-4)(x+4)$. 48. $(x-2)(x-3)(x+7)^2$.
49. $(x-2y)(x-4y)(x+6y)(x-8y)$. 50. $(x+7)(2x-3)(3x-1)(3x+1)$.
51. $x(x-1)(x+1)(2x+3)(5x-2)$. 52. $(x+5a)(x^3-125a^3)$.
53. $xy(x-y)(x^4+x^2y^2+y^4)$.
54. $(x-1)(x+1)(x-3)(x^2-3)(x^2+x-3)$; vide Ex. 43, 44, Chap. VII.
55. $24x^2y^2(x^4-y^4)(x^2+xy+y^2)$. 56. $(x-a)(x-b)(x-c)$; vide Art. 33.
57. $(x-a)(x-b)(x+c)$. 58. $(x+a)(x-b)(x-c)$.
59. $(x+y+z)(y+z-x)(z+x-y)(x+y-z)$. 60. x^8-y^8.
61. $(a^2-b^2)(a^2-9b^2)$. 62. $(a-b)^2(a-2b)(a+4b)$.
63. $(a-1)(a-2)(a+3)(a+4)$. 64. $(a+2b)(a-b)(a^3+b^3)$.
65. $(x-1)^2(x+2)(x-3)(x+3)$. 66. $(x+1)^2(x+2)(x-4)$.
67. $(x+2a)(x+5a)(x^2-2ax+2a^2)$. 68. $(x-2y)(x-4y)(x^2+2xy+3y^2)$.
69. $(x+1)(2x-3)(3x+1)(4x-3)$. 70. $(x-1)(2x-3)(3x^2+4x+5)$.
71. $(7x+12y)(9x+11y)(x^2-4xy+2y^2)$. 72. $(x-1)^3(x+1)(x+3)$.
73. $(x+1)(x+2)(x^2+2x+3)(x^2-5x+1)$.
74. $(x-y)^2(x+y)(x+3y)(x^2+4xy+2y^2)$.
75. $(x+1)(3x-1)(2x^2-x+2)(3x^2+2x+1)$.
76. $12x^3(x-2a)^2(x+3a)(3x+2a)$. 77. $6xy(x+3y)(2x+y)(2x-y)^2$.
78. $(x-1)(x-2)(x-3)(x^2+2x+3)$. 79. $(x+2a)^2(x-2a)(x+4a)(x-4a)$.
80. $(x-3y)^2(x-4y)(x-5y)(x^2+xy+2y^2)$.

CHAPTER IX.

1. $\frac{a}{8bc^2}$. 2. $\frac{x^2}{6y^2}$. 3. $\frac{yz}{3x^2}$. 4. $\frac{ac}{13xy^2z^2}$. 5. $\frac{a-b}{a+b}$.

6. $\frac{1}{a-b}$. 7. $\frac{a+b}{a^2-ab+b^2}$. 8. $\frac{a(a-1)}{a^2-a+1}$. 9. $\frac{a+b-c}{a-b+c}$.

10. $\frac{a-b}{a-b+c}$. 11. $\frac{a+b}{a+b+2c}$. 12. $\frac{a+b}{a-b}$. 13. $\frac{a^2+b^2}{(a-b)^2}$.

14. $\frac{b(a+b)}{a^2+ab+b^2}$. 15. $\frac{ab}{a^2-b^2}$. 16. $\frac{x-7}{x-5}$. 17. $\frac{x-6}{x-10}$.

18. $\frac{x-4}{x+4}$. 19. $\frac{5x}{5-x}$. 20. $\frac{x+b}{x-c}$. 21. $\frac{x-a}{x-c}$. 22. $\frac{x-a}{x-b}$.

23. $\frac{x-a}{x+7a}$. 24. $\frac{x+9y}{x-4y}$. 25. $\frac{ax-1}{2x-a}$. 26. $\frac{x+3}{x-3}$. 27. $\frac{3x-4}{4x-3}$.

28. $\frac{3x-2}{5x-3}$. 29. $\frac{4x+y}{3x+y}$. 30. $\frac{bx+a}{ax+b}$. 31. $\frac{abx-1}{cx-1}$.

32. $\frac{(x+2)^2}{x^2+2x+4}$. 33. $\frac{x^2+x-3}{x^2+x+1}$. 34. $\frac{x-y}{3y}$. 35. 2.

36. $\frac{x+y}{x}$. 37. $\frac{x^2(x^2-y^2)}{x^4-x^2y^2+y^4}$. 38. $\frac{x+y}{x-y}$. 39. $\frac{x+a}{x-a}$.

40. $\frac{x+y-z}{x+y+z}$. 41. $\frac{x+3}{x-3}$. 42. $\frac{x-4}{x+6}$. 43. $\frac{x-1}{x+1}$.

44. $\frac{x}{x^2+x+1}$. 45. $\frac{x^2-x+1}{x^2-1}$. 46. $\frac{ax}{x^2+a^2}$. 47. $\frac{x+1}{x+2}$.

48. $\frac{x-y-z}{x+y-z}$. 49. $\frac{x+y-z}{x-y-z}$. 50. $\frac{1}{x^2-y^2}$. 51. $\frac{a-3b}{a^2-3ab+b^2}$.

52. $\frac{2a+b}{a+7b}$. 53. $\frac{a+b+c}{a-b+c}$. 54. $\frac{x-1}{x+1}$. 55. $\frac{x-1}{x+2}$.

56. $\frac{(x-a)(x-2a)}{(x-3a)(x-4a)}$. 57. $\frac{2x+3y}{3x+2y}$. 58. $\frac{bc-ca+ab}{bc+ca+ab}$.

59. $\left(\frac{a-b}{a+b}\right)^2$. 60. $\frac{1}{x+3}$. 61. $\frac{x^2-3x+1}{(x-1)(x-2)}$. 62. $\frac{x+2y}{x-2y}$.

63. $\frac{1}{x-2a}$. 64. $\frac{x^4-y^4}{x^4+y^4}$. 65. $\frac{1}{x+y+z}$.

In the following twenty-five examples the common denominator is written first and the numerators in order after the colon.

66. $b^3 : ab^2, a^3$. 67. $b^2c^4 : ac^3, ab^3$. 68. $12a^3 : 8a^2, 9a, 10$.

69. $b(a^2-b^2) : b(a+b), a(a-b), b^2$.

70. $ac(b^2-c^2) : a^2(b-c), abc, c^2(b+c)$.

71. $a^6-b^6 : a(a^3+b^3), b(a^3-b^3), ab(a-b)(a+b)$.

72. $(a-b+c)(a+b-c)(a-b-c) : a-b-c, a+b-c$.

73. $(a+c)^2-b^2 : a-b+c, a+b+c, c$.

74. $125a^3+27b^3 : 25-15ab+9b^2,\ a^2,\ a(5a+3b)$.

75. $8a^3-343b^3 : 4a^2+14ab+49b^2,\ a(2a-7b),\ ab$.

76. $(x-a)^2(x+a)^2 : (x-a)^2(x+a),\ 2a(x-a)^2,\ 3(x-a)(x+a)^2,\ 4a(x+a)^2,\ 5a(x^2-a^2)$.

77. $(x-y)^2(x^2+xy+y^2) : x^3-y^3,\ (x-y)^2,\ x^2+xy+y^2,\ x-y$.

78. $(x-a)(x+b)(x+c) : x-a,\ x+b,\ x+c$.

79. $(x^2-a^2)(x-b) : x-b,\ x+a,\ x-a$.

80. $(x-1)(x-3)(x+7) : x+7,\ x-1,\ x-3$.

81. $(x^2-9)(x^2-144) : (x^2-9)(x-12),\ (x-3)^2(x+12),\ x(x^2-9)$.

82. $(x+2a)(2x-a)(3x+2a) : 3x+2a,\ 2x-a,\ x+2a$.

83. $(x-2y)^2(x^2+2xy+4y^2) : x^2-4y^2,\ (x-2y)^2,\ x^2+2xy+4y^2$.

84. $(x-1)^2(x^2-3x-3) : x-1,\ x^2-3x-3,\ (x-1)^2$.

85. $(x^2-y^2)(x^3+y^3) : x^3+y^3,\ (x-y)(x^2-xy+y^2),\ (x-y)(x+y)^2,\ (x-y)^2(x+y)$.

86. $(a+b)^2(a-6b) : a+b,\ a-b$.

87. $(a-2b)(a+3b)(a-4b)(a-5b) : a-5b,\ a-4b$.

88. $(x^2-4)(x^2+x+1) : x+2,\ x-2$.

89. $(x-y)(x^3+2x^2y+2xy^2+2y^3) : (x-y)^2,\ x^2+y^2$.

90. $(x-a)^3(x+2a)(x+3a) : x+2a,\ x+3a$.

91. $\frac{2a}{a^2-b^2}$. 92. $\frac{1}{a-b}$. 93. $\frac{a}{a^2-b^2}$. 94. $\frac{4ab}{a^2-b^2}$.

95. 1. 96. $\frac{a+b}{ab}$. 97. $\frac{a-b}{a+b}$. 98. 0. 99. $\frac{2}{x+1}$.

100. $\frac{60}{(x^2-4)(x^2-9)}$. 101. 1. 102. $\frac{2}{x-y}$. 103. 0.

104. $\frac{xy}{x^3+y^3}$. 105. $\frac{1}{x^3-1}$. 106. $\frac{1}{x^3+1}$. 107. 0.

108. $\frac{1}{x-a}$. 109. $\frac{a}{x-a}$. 110. $\frac{3c-a-2b}{(x-a)(x-b)(x-c)}$.

111. $\frac{2(x+3)}{(x-2)(x-3)(x+8)}$. 112. $\frac{2}{(x-11)(x+12)(x-13)}$.

113. $\frac{2x-1}{(x+1)(x-2)(2x+1)}$. 114. $\frac{5}{(x+4)(2x-3)(3x-2)}$.

115. $\left(\frac{4}{x^2-4}\right)^2$. 116. $\frac{18}{(x-1)(x-2)(x+5)}$. 117. $\frac{15}{(x+2)(x-3)^2}$.

118. $\frac{2}{(x-1)(x^3-4x+3)}$. 119. 1. 120. 0.

121. $\frac{bc}{6a^2}$. 122. 1. 123. $\left(\frac{b}{a-b}\right)^2$. 124. $a-b$.

125. $\frac{1}{a^2-b^2}$. 126. 1. 127. $\frac{1}{x^2+y^2}$. 128. 1.

129. 1. 130. $\frac{1}{(x-a)(x-b)(x+c)}$. 131. 1. 132. 1.

133. 1. 134. $\frac{x}{x+2}$. 135. 1. 136. 1. 137. 1.

138. 1. 139. $\frac{x-y}{x+y}$. 140. 1. 141. $\frac{1}{a-b+c}$.

142. $\frac{a+b}{a-b}$. 143. $\frac{1}{x^2-4}$. 144. $\left(\frac{x-1}{x-3}\right)^2$. 145. 1.

146. $\frac{(x+1)(x+2)}{(x-1)(x-2)}$. 147. 1. 148. $\frac{x-3}{x-4}$. 149. $\frac{x+2}{x-2}$.

150. 1. 151. $\frac{3ab}{8c^2}$. 152. $\frac{5abc}{7xyz}$. 153. $a+b$.

154. $\frac{1}{a^2-b^2}$. 155. $\frac{1}{a^2-b^2}$. 156. $a\left(x+\frac{1}{a}\right)$.

157. $\frac{x}{a}+\frac{a}{x}$. 158. $x-1+\frac{1}{x}$. 159. $\frac{x+y}{x}$.

160. $\frac{x-y}{y}$. 161. $\frac{x}{y}+1+\frac{y}{x}$. 162. 1. 163. 1.

164. 1. 165. 1. 166. 1. 167. $\frac{x-3y}{x+3y}$.

168. $\frac{1}{y+z-x}$. 169. $\frac{x+y-z}{2(x-y-z)}$. 170. 1.

171. $\frac{(x+3y)(x+4y)}{(x-3y)(x-4y)}$. 172. 1. 173. $x+1$.

174. $\frac{5x-3}{3x-2}$. 175. $\frac{1}{x^2-x+1}$. 176. $\frac{x^2+x-1}{x^2+2x-1}$. 177. $\frac{x-5}{x-7}$.

178. $\frac{x-5}{x-4}$. 179. $\frac{x+2}{x-2}$. 180. $\frac{x-1}{x+1}$. 181. $\frac{1}{x+1}$.

182. 0. 183. 0.

198. Square. 199. $a-\frac{1}{a}=0$, $\therefore a^n-\frac{1}{a^n}=0$; square.

200. $\left(a-\frac{1}{a}\right)^3=0$; $\therefore a=\frac{1}{a}$.

CHAPTER X.

1. $a^4x^8-2ax^4y^3+\frac{y^6}{a^2}$. 2. $1-2a+3a^2-2a^3+a^4$.

3. $\frac{x^2}{a^2}+\frac{a^2}{x^2}+2\left(\frac{x}{a}-\frac{a}{x}\right)-1$. 4. $4x^2+\frac{y^2}{4}+16z^2-4yz+16zx-2xy$.

5. $\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}-2\left(\frac{y}{x}-\frac{z}{y}+\frac{x}{z}\right)$. 6. $\frac{x^2z^2}{x^2}+\frac{z^2x^2}{y^2}+\frac{x^2y^2}{z^2}+2(x^2+y^2+z^2)$.

7. $(1+x^2+x^4)^2=1+2x^2+3x^4+2x^6+x^8$.

8. $(1-x^2+x^4)^2=1-2x^2+3x^4-2x^6+x^8$.

9. $3a-2\sqrt{2a^2-ax-x^2}$. 10. $2x+2\sqrt{x^2-(y-a)^2}$.

11. $a^3+\frac{1}{a^3}+3\left(a+\frac{1}{a}\right)$.

12. $1+6a+12a^2+8a^3$.

13. $8x^3-4x^2y+\frac{2}{3}xy^2-\frac{y^3}{27}$.

14. $27x^6-108x^4y+144x^2y^2-64y^3$.

15. $1+3a-5a^3+3a^5-a^6$.

16. $(1+a^3)^3=1+3a^3+3a^6+a^9$.

17. $(1+x^2+x^4)^3=1+3x^2+6x^4+7x^6+6x^8+3x^{10}+x^{12}$.

18. $(1+x^4)^3=1+3x^4+3x^8+x^{12}$.

19. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.

20. $x^3-\frac{1}{x^3}-3\left(x^2+\frac{1}{x^2}\right)+5$.

21. $2(a^2+4b^2+9c^2+6ca)$, $8b(a+3c)$.

22. $2a(a^2+3b^2)$, $2b(3a^2+b^2)$.

23. $2(a^4+6a^2b^2+b^4)$, $8ab(a^2+b^2)$.

24. $2(a^6+15a^4b^2+15a^2b^4+b^6)$, $4ab(3a^4+10a^2b^2+3b^4)$.

25. $2(a^2+b^2+c^2+d^2+2bc+2ca+2ab)$, $4d(a+b+c)$.

26. $(x^2-a^2)^4=x^8-4a^2x^6+6a^4x^4-4a^6x^2+a^8$.

27. $\frac{x^4}{a^4}+\frac{a^4}{x^4}+2\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}\right)+6$.

28. $1-5x+10x^2-10x^3+5x^4-x^5$.

29. $x^6+\frac{1}{x^6}-6\left(x^4+\frac{1}{x^4}\right)+15\left(x^2+\frac{1}{x^2}\right)-20$.

30. $x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5+28x^2y^6+8xy^7+y^8$.

31. $6a^2$.

32. $12x^3y^4z^5$.

33. $\frac{11x}{13y^2}$.

34. $3a+7b$.

35. $4a^2b^2-5c^2$.

36. $10x+11y$.

37. $(x+1)(x-2)(x-3)$.

38. $\frac{2x-9}{2x-15}$.

39. $\frac{3ax-4b}{3ax+4b}$.

40. $\frac{1}{x+2}$.

41. $\frac{17x+2y}{19x^2+3y^2}$.

42. $\frac{ax}{by}-\frac{by}{ax}$.

43. $\frac{x}{2}+\frac{2}{3}$.

44. $\frac{x}{3}-\frac{a}{2}$.

45. $4a-\frac{b}{4}$.

46. $x-y-a$.

47. $x^3-2a^2y+\frac{ay^2}{2}$; given expression is

$$x^6-2x^3\left(2a^2y-\frac{ay^2}{2}\right)+4a^4y^2-2a^3y^3+\frac{a^2y^4}{4}.$$

48. $3a-\frac{2b}{3}+\frac{c}{2}$; given expression is $9a^2-6a\left(\frac{2b}{3}-\frac{c}{2}\right)+\frac{4b^2}{9}-\frac{2bc}{3}+\frac{c^2}{4}$.

49. x^2-x-a; given expression is $x^4-2x^3+x^2-2a(x^2-x)+a^2$.

50. $x^2-yz+\frac{y^3}{z}$; given expression is $x^4-2x^2\left(yz-\frac{y^3}{z}\right)+y^2z^2-2y^4+\frac{y^6}{z^2}$.

51. $\frac{a^2}{bc}-\frac{b}{c}-\frac{c}{b}$.

52. x^a-2x^b.

53. a^2x^2+ax+y; given expression is $a^4x^4+2a^3x^3+a^2x^2+2y(a^2x^2+ax)+y^2$; arrange the expressions in Ex. 54—57 as in this example and in Ex. 47—50, vide Art. 80.

54. ax^2-bx-b^2.

55. $\left(1+\frac{1}{2n}\right)x-\frac{m}{x}$.

56. $\frac{x}{y}+\frac{y}{x}-2m$.

57. $x^3+\frac{a}{x}-\frac{1}{x^2}$. 58. $\frac{x}{y}-\frac{y}{z}+\frac{z}{x}$. 59. $\frac{x}{y}+\frac{y}{x}-1$.

60. $\frac{x}{y}-\frac{y}{x}-2$. 61. $a^2-3ab+b^2$. 62. $1-2a-a^2$.

63. $a^2+2ab+3b^2$. 64. $a^2-a+\frac{1}{2}$. 65. $(1+x)^2$. 66. $x^2-3x-\frac{2}{x}$.

67. $x^4-2x^3y+\frac{y^5}{2x}$. 68. $x^2-x-\frac{1}{x}$. 69. x^3-2x^2+1.

70. $ax^3-x^2-\frac{1}{a}$. 71. $23x^2-x+\frac{17}{x}$. 72. $1-a+a^2-a^4$.

73. $\frac{1}{2a}+1-a-a^2$. 74. $a^2-2ac+bc$. 75. $\frac{2x}{3y}-\frac{2y}{z}+\frac{3z}{4x}$.

76. $3ab-4c^2$. 77. $2x+y$. 78. $x-\frac{1}{x}$. 79. x^2-5. 80. $29-a$.

81. x^2-2x+y; given expression is

$$y^3+3y^2(x^2-2x)+3y(x^4-4x^3+4x^2)+x^6-6x^5+12x^4-8x^3;$$

work this example and the two following by the method of Art. 86.

82. x^2-ax+b. 83. $x-3y+z$. 84. $x-\frac{1}{x}+1$. 85. $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$.

86. $x^2-2xy-y^2$. 87. $1-x-x^2$. 88. $a^3-a^2-\frac{1}{a}$. 89. $\frac{x^2}{3}+x-\frac{3y}{x}$.

90. $1-x+x^3-x^4$. 91. $x-y$. 92. $2x^2-3a^2$. 93. $4a-5b$.

94. $\frac{a}{b}+\frac{b}{a}$. 95. $1+x-x^3$. 96. $x+\frac{1}{x}$. 97. $2-\frac{x}{2}$.

98. $x-\frac{1}{x}$. 99. x^2+2. 100. $x-1$.

CHAPTER XI.

1. 5. 2. 2. 3. -6. 4. -2. 5. 2. 6. 3. 7. -8.
8. $-\frac{1}{2}$. 9. 2. 10. 7. 11. $\frac{2}{3}$. 12. -1. 13. $\frac{6}{5}$. 14. 12.
15. 5. 16. -21. 17. 8. 18. 1. 19. 0. 20. 3.
21. -10. 22. $-\frac{1}{2}$. 23. 40. 24. 18. 25. 4. 26. 32.
27. 19. 28. 0. 29. -19. 30. 8. 31. $-\frac{1}{6}$. 32. -1.
33. $\frac{1}{3}$. 34. 7. 35. 0. 36. $\frac{1}{10}$. 37. 17. 38. 7. 39. 10.
40. -1. 41. $\frac{4}{3}$. 42. $\frac{4}{5}$. 43. $-\frac{10}{7}$. 44. 0. 45. 2. 46. $\frac{3}{35}$.
47. $-\frac{7}{5}$. 48. $-\frac{1}{3}$. 49. $-\frac{17}{2}$. 50. 22. 51. 0. 52. $\frac{3}{2}$.
53. -3. 54. -3. 55. $\frac{6}{25}$. 56. $\frac{31}{41}$. 57. $\frac{1}{21}$. 58. 4.
59. 2. 60. 0. 61. 0. 62. 1. 63. 26. 64. 4. 65. $\frac{7}{2}$.
66. -6. 67. 2. 68. $\frac{1}{2}$. 69. $-\frac{1}{2}$. 70. 16. 71. -10.
72. 25. 73. 13. 74. 33. 75. 1. 76. 6. 77. $\frac{1}{2}$. 78. $23a$.
79. $a+b$. 80. a^2-ab+b^2. 81. $a+b$. 82. $a+7b$. 83. $a-2b$.

84. $a-4b$. 85. $4a+3b$. 86. $5a$. 87. $\frac{a}{b}$. 88. 1. 89. 0.

90. 1. 91. 0. 92. $\frac{ab}{a+b}$. 93. 0. 94. $\frac{a}{2}$. 95. $\frac{ab}{a+b}$.

96. $\frac{a^2+b^2}{a+b}$. 97. $(a+b)^2$. 98. $bc+ca+ab$. 99. $\frac{a+b+c}{2}$.

100. bc. 101. $\frac{a+b}{2}$. 102. $\frac{2ab}{a+b}$. 103. 0. 104. $\frac{ab}{a^2+b^2}$.

105. $\frac{ab}{a+b}$. 106. $\frac{a+b+c}{2}$. 107. $\frac{3ab}{a+b}$. 108. $a+b+1$.

109. $\frac{a+b+1}{2}$. 110. $-\frac{b+c}{2}$. 111. $\frac{2(bc-a^2)}{b+c-2a}$. 112. 9.

113. 169. 114. $\frac{1}{6}$. 115. 1. 116. 0. 117. $\frac{a}{4}$. 118. 0.

119. a. 120. $\frac{5a}{4}$. 121. 5, 4. 122. 17, 70. 123. 3, -7.

124. $\frac{1}{4}$, -2. 125. $\frac{1}{3}$, $\frac{1}{2}$. 126. 12, 16. 127. $\frac{1}{3}$, $-\frac{1}{10}$.

128. 24, 35. 129. 12, 3. 130. 0, 0. 131. $\frac{1}{4}$, $\frac{1}{6}$. 132. -2, -3.

133. 5, -5. 134. 7, 1. 135. $\frac{9}{2}$, $\frac{1}{3}$. 136. 1, 1. 137. 13, 15.

138. 2, 1. 139. $\frac{1}{3}$, $\frac{1}{14}$. 140. 4, 10. 141. 1, $\frac{1}{2}$. 142. $-\frac{1}{2}$, $-\frac{1}{3}$.

143. 3, 4. 144. $\frac{1}{12}$, $\frac{1}{10}$. 145. $\frac{1}{4}$, -4. 146. $\frac{1}{6}$, 3. 147. $\frac{1}{5}$, 6.

148. -1, -2. 149. 3, -1. 150. 5, -1. 151. 5, 4.

152. $\frac{1}{3}$, $-\frac{2}{3}$. 153. b, a. 154. a, b. 155. a, b. 156. $a+b$, 0.

157. $a-b$, $a+b$. 158. $\frac{a+b}{2}$, $\frac{a-b}{2}$. 159. $(a+b)^2$, $(a-b)^2$.

160. $a+b$, $a-b$. 161. a^2+ab+b^2, a^2-ab+b^2. 162. 0, 0.

163. a, b. 164. $\frac{a^2-ab+b^2}{a-b}$, $\frac{ab}{a-b}$. 165. $a-b$, $a+b$.

166. $\frac{2ab}{a^2+b^2}$, $\frac{2ab}{a^2-b^2}$. 167. $\frac{ab}{a+b}$, $\frac{ab}{a-b}$ 168. a, b. 169. $2a$, a.

170. $\frac{5a}{4}$, $\frac{4a}{5}$. 171. 1, 2, 3. 172. 9, -2, -3. 173. $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{2}$.

174. 0, 3, 4. 175. -2, 1, 2. 176. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{8}$. 177. 2, 2, 2.

178. 6, 3, 4. 179. $\frac{3}{4}$, $\frac{1}{2}$, 1. 180. 12, 6, 8. 181. $\frac{10}{3}$, $\frac{3}{2}$, $\frac{4}{9}$.

182. $\frac{1}{6}$, $\frac{1}{9}$, 3. 183. 4, -4, 1. 184. 23, 47, 67. 185. 0, 10, 4.

186. 1, 2, 3. 187. 1, 1, -1. 188. 3, 6, 9. 189. 3, 4, 6.

190. a, $\frac{b}{2}$, $\frac{c}{3}$. 191. a, $2b$, $3c$. 192. $2a$, b, 0.

193. $\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}=\frac{abc}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$.

194. $\frac{1}{b}+\frac{1}{c}-\frac{1}{a}$, $\frac{1}{c}+\frac{1}{a}-\frac{1}{b}$, $\frac{1}{a}+\frac{1}{b}-\frac{1}{c}$. 195. a, a, a.

196. $b+c-2a$, $c+a-2b$, $a+b-2c$.

197. $a(a+b+c)$, $b(a+b+c)$, $c(a+b+c)$. 198. $\frac{b+c}{3}$, $\frac{c+a}{3}$, $\frac{a+b}{3}$.

199. $\frac{b}{2}$, $\frac{a}{2}$, $\frac{ab}{2c}$. 200. $x=1$, $y=2$, $z=3$, $u=-1$.

CHAPTER XII.

1. 35, 40. 2. 8, 16. 3. 8, 9. 4. 11, 12, 13. 5. 15, 8.
6. 48. 7. 14, 20. 8. 144. 9. 17, 18. 10. 6. 11. 12, 15.
12. 12, 48. 13. 3, 4, 5. 14. 120. 15. 37. 16. 4095.
17. 20, 16, 12, 8. 18. 120, 121. 19. 75, 125. 20. 3024.
21. 31, 57. 22. 24, 7. 23. 9, 5. 24. 27, 33, 10, 90.
25. £132, £308, £693. 26. £30, £20, £5. 27. 7, 26.
28. 15; Rs. 3. 6 as. 29. £250, £200, £150. 30. 12, 20.
31. 972 bighas. 32. 44. 33. 34, 18. 34. 4, 12. 35. £3960.
36. Rs. 7, Rs. 3. 37. £1300, £700, £600.
38. Rs. 52. 13 as. 3 p., Rs. 47. 2 as. 9p. 39. £24, £20, £16. 40. 6, 15.
41. Re. 1. 12 as., Re. 1. 4as. 42. 210. 43. £6600. 44. Rs. 60, Rs. 30.
45. $2\frac{1}{2}$ doz., $1\frac{1}{2}$ doz. 46. Rs. 3000, Rs. 7000. 47. £360, £240.
48. £1500; £45, £33. 49. Rs. 10296. 50. 1*s*. 1*d*. 51. £21.
52. 1*s*. $1\frac{3}{4}$*d*. 53. Rs. 900. 54. 2*s*. 2*d*. 55. 48. 56. 7 ft. 6 in.
57. 36 sq. ft. 58. 3 ft., 9 ft. 59. 12, 9, 7, 6, inches.
60. 15, 12, yds. 61. 120 miles. 62. 90 miles.
63. 22 miles; 24, 20, miles an hour. 64. 69 miles.
65. 368, 366, yds. per minute; $191\frac{2}{15}$ yds. 66. 160, 140, miles per day.
67. 60. 68. 14, 12, miles per hour. 69. 20. 70. 14, 8.
71. 36, 64, 100. 72. 6, 2, 12. 73. 3 hours 44 min. 74. 39, 46.
75. 24 min. 76. $3\frac{12}{17}$ hours. 77. $16\frac{4}{11}$ min. past 3.
78. $10\frac{10}{11}$, $54\frac{6}{11}$, min. past 6. 79. $1\frac{1}{59}$, $1\frac{1}{3628}$, min. past 12. 80. 4 seers.
81. 22. 82. 60. 83. 21 days; 1421 miles.
84. 21 days; 972 miles. 85. $5\frac{5}{11}$ min. past 7. 86. 25 min. past 5.
87. 1 min., 2 min. 88. 1331, 1000. 89. 1000. 90. 55.
91. 696. 92. 15 doz., 24 doz. 93. Re. 1. 4 as. 6p. 94. $\frac{1}{5}$.
95. 2508 ft. 96. Jan. 5th, 1894, 7 P.M.; 1530 miles. 97. 63, 84, days.
98. 32, 68. 99. 400, 300. 100. 1*s*. 2*d*. 101. 29, 37.
102. 10, 8. 103. 59, 94. 104. 15, 60. 105. 9, 12. 106. $\frac{9}{17}$.
107. $\frac{3}{26}$. 108. $\frac{7}{13}$. 109. 78. 110. 63. 111. 75, 57.
112. 84. 113. Rs. 125, Rs. 75. 114. Rs. 5. 8 as., Rs. 4. 115. 5*s*., 2*s*.
116. 20, 10, 4. 117. 48, 14. 118. 13, 6, 7. 119. 3*s*. 6*d*., 3*s*.
120. Rs. 8; 12, 6, 4. 121. 8 : 7; $\frac{16}{21}$ quart, $\frac{2}{3}$ quart. 122. 80, 24.
123. 49, 33. 124. 26. 125. 14, 8. 126. 11; 16, 13.
127. 10 days, 2 days; Feb. 21st, 29th. 128. 30, 24, days.

129. 5000 gall., 4000 gall. 130. 4, 3, miles an hour.
131. 16 miles; 8 miles an hour. 132. 11 : 1.
133. In each candy 2 pharas of best rice, 6 pharas of inferior rice.
134. 4 : 11. 135. 48 lbs., 40 lbs. 136. £1200, £1800.
137. Rs. 5000, Rs. 7000. 138. Rs. 1200, Rs. 1000. 139. £25; 1*s.* 8*d.*
140. £1400; 1*s.* 2*d.* 141. £132; 1*s.* 10*d.*; Rs. 1440, Rs. 1980, Rs. 2112.
142. 3, 5, miles an hour. 143. 20 miles; 6 miles an hour.
144. 1, $1\frac{1}{2}$, miles an hour. 145. 5, 3, hours. 146. 119, 120, miles.
147. 24 miles an hour; 50 miles. 148. 45, 30, miles an hour.
149. 12 miles an hour, 165 miles. 150. 51, 50, miles. 151. 3, 4, ft.
152. 24; Rs. 7. 153. Rs. 900; Rs. 1000.
154. 4 lakhs, $2\frac{1}{2}$ lakhs, 1 lakh. 155. 121 yds., 40 yds.; 1 acre.
156. 40, 45, 48, days. 157. 5000; Rs. 5. 158. 8, 10, miles an hour.
159. 100; 44, 56. 160. 21, 22. 161. 1, 2, 3. 162. 4, 5, 8.
163. 3, 5, 13. 164. 1, 2, 3. 165. 5, 4, 3. 166. 13, 12, 5.
167. 65, 52, 39. 168. 405. 169. 105, 405. 170. 108, 801.
171. 235. 172. 9, 7, 29. 173. 3345. 174. 738. 175. $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{12}$.
176. Re. 1. 4 as., 14 as., 3 as. 177. 9*s.*, 8*s.*, 4*s.* 178. 20, 50, 80.
179. Gram Rs. 22, corn Rs. 20, rice Rs. 40, per candy.
180. £300, £280, £220. 181. 2 oz., 4 oz., 6 oz. 182. 5, 4, $4\frac{1}{2}$, days.
183. Rs. 2. 8 as., Rs. 2. 2 as., Re. 1. 12 as. 184. 5, 6, 8, annas.
185. 20, 30, 24, days. 186. 6000, 3000, 2000, gallons.
187. (1) 1 hour, (2) 2 hours, (3) water flows out exactly as fast as it flows in.
188. Each lb. contained $\frac{5}{12}$, $\frac{11}{24}$, $\frac{1}{8}$ of the teas respectively.
189. 1, 2, 9. 190. Length 60 ft., breadth 40 ft., height 30 ft.; Rs. 1500.
191. £21000, £8000, £6000. 192. 15, 5, 30, 24. 193. 32, 22, 28.
194. £12, £1. 5*s.*, £1. 7*s.* 6*d.* 195. 84, 58, 21.
196. Rs. 800, Rs. 900, Rs. 1000. 197. £40, £30; 1*s.* 4*d.*, 1*s.* 3*d.*
198. 6, 5, 12, miles an hour. 199. 20, 12, 15, miles an hour.
200. 44, 11; 48, 13; 46, 96; 50, 100.

CHAPTER XIII.

1. 1, 4. 2. 12, −9. 3. −18, −10. 4. 13, −15. 5. −1, $\frac{4}{3}$.
6. 3, $-\frac{12}{7}$. 7. $\frac{2}{3}$, $-\frac{5}{4}$. 8. $\frac{7}{2}$, $\frac{2}{9}$. 9. 10, −2. 10. 6, −20.
11. 3, 8. 12. 7, $\frac{23}{5}$. 13. 1, $\frac{25}{43}$. 14. 6, $-\frac{2}{5}$. 15. $\frac{3}{2}$, $-\frac{5}{9}$.
16. $\frac{1}{2}$, $-\frac{2}{7}$. 17. $\frac{3}{7}$, $\frac{3}{7}$. 18. $\frac{7}{12}$, $-\frac{1}{3}$. 19. 4, $-\frac{1}{4}$. 20. 3, $\frac{1}{3}$.
21. 2, −1. 22. 1, 1. 23. $2\pm\sqrt{5}$. 26. 0, −23. 27. $\frac{1}{2}$, $-\frac{3}{5}$.
28. 1, $-\frac{1}{2}$. 29. $2(1\pm\sqrt{2})$. 30. −3, 10. 31. 2, −3. 32. 3, $\frac{1}{2}$.
33. 1, $-\frac{10}{7}$. 34. $\frac{1}{6}$, $-\frac{7}{17}$. 35. ± 3. 36. 4, $-\frac{7}{3}$. 37. 2, 3.
38. 1, $-\frac{11}{6}$. 39. 1, 6. 40. 16, −5. 41. −1, $-\frac{27}{5}$.

42. $90, -10$. 43. $1, 2$. 44. $2, 5$. 45. $4, 4\frac{3}{22}$. 46. $2, -\frac{1}{4}$.

47. $6, \frac{10}{3}$. 48. $12, \frac{1}{12}$. 49. $5, \frac{1}{5}$. 50. $8, -9$.

51. $16a, 17a$. 52. $11a, -14a$. 53. $\frac{4a}{7}, -\frac{a}{2}$. 54. $\frac{4a}{3}, -\frac{12a}{13}$.

55. $3a, \frac{6a}{5}$. 56. $a, \frac{a}{3}$. 57. $0, -3a$. 58. $a, \frac{2a}{3}$.

59. $a, -\frac{a}{7}$. 60. $2a, \frac{2a}{3}$. 61. $a-b, a+b$. 62. $-a, 2a-b$.

63. $\frac{b}{a}, -\frac{a}{b}$. 64. $\frac{a}{a-b}, -\frac{b}{a+b}$. 65. $a, -b$. 66. $\frac{b \pm a}{5}$.

67. $1, \frac{a-b}{b-c}$. 68. $-1, \frac{a+b+c}{a}$. 69. $\frac{a+b+c}{a \pm (b-c)}$.

70. $\frac{a}{b+c}, \frac{b+c}{a}$. 71. $1, a^2$. 72. $3a, -\frac{a}{3}$. 73. $\frac{a}{4}, -4a$.

74. $a, -\frac{7a}{10}$. 75. $\frac{1-m}{1+m}a, \frac{1+m}{1-m}a$. 76. $-a, -a$.

77. $0, 6a$. 78. $a \pm \frac{2}{a}$. 79. $3a, 4a$. 80. $a, \frac{m+n}{2}a$.

81. $\frac{a^2}{b^2}, -\frac{2a^2}{b^2}$. 82. $0, -\frac{a^2+b^2}{a+b}$. 83. $a+b, \frac{2ab}{a+b}$.

84. $0, -\frac{a+b}{2}$. 85. $a+b, \frac{a+b}{2}$. 86. $0, -\frac{a^3+b^3}{b^2}$.

87. $a+b, \frac{a+b}{a^2+b^2}ab$. 88. $\frac{a+b}{2}, \frac{a+b}{a^2+b^2}ab$. 89. $-(a+b), -(a+b)$.

90. $a+b, \frac{a+b}{a^2+b^2}ab$. 91. $c \pm \sqrt{ab - c(a+b-c)}$.

92. $c, -\frac{a+b+2c}{bc+ca+2ab}ab$. 93. $c, -\frac{a+b}{2}$. 94. $a+b \pm c$.

95. $c, a+b+c$. 96. $c, \frac{ab}{c}$. 97. $4c, a+b+2c$.

98. $2(a+b), 2(a+b+c)$. 99. $a+b+c, a+b+2c$. 100. $a+b+c, -\frac{2ab}{a+b}$.

101. $\pm 2, \pm 3$. 102. $\pm\sqrt{2}, \pm\sqrt{3}$. 103. $\pm 5, \pm\frac{1}{2}$.

104. $\pm\frac{4}{3}, \pm\frac{7}{6}$. 105. $\pm a, \pm 4a$. 106. $\pm\sqrt{a}$.

107. $-1, 1 \pm \frac{\sqrt{-3}}{2}$; $2, -1 \pm \sqrt{-2}$.

108. $1, \frac{-1 \pm \sqrt{-3}}{2}$; $3, \frac{-3 \pm 3\sqrt{-1}}{2}$. 109. $\pm\frac{ax}{b}, \pm\frac{bx}{a}$.

110. $\pm\frac{1}{c}, \pm\frac{c}{ab}$. 111. $1, \frac{-1 \pm \sqrt{-3}}{2}$. 112. $1, \pm\sqrt{-1}$.

113. $\frac{3 \pm \sqrt{5}}{2}$. 114. $-a$. 115. $\pm 1, \pm 2, \pm\frac{1}{2}$. 116. $3, -\frac{10}{7}$.

117. $3, \frac{6}{5}$. 118. $\frac{11 \pm \sqrt{13}}{6}$. 119. $2a, -\frac{3a}{2}$. 120. $0, 8$. 121. $0, c$.

122. $1, -2; 2, -3$. In this example put $x^2+x=y$, and employ a similar artifice in the following fifteen examples.

123. $2, -1$. 124. $1, -2; 0, -1$. 125. $\pm 2, \pm 3$.

126. $2, \frac{1}{2}; 1, \frac{3}{2}$. 127. $1, -\frac{1}{2}; 2, -\frac{3}{2}$. 128. $2, 4, 0$.

129. $-2, \frac{1}{3}; 1, -\frac{8}{3}$. 130. $\frac{3}{2}, -\frac{2}{3}$. 131. $1, \frac{-1 \pm \sqrt{-3}}{2}; 2, -1 \pm \sqrt{-2}$.

132. 1. 133. $8, -1$. 134. $2, -5$. 135. $0, 1; \frac{3}{2}, -\frac{1}{2}$.

136. $\frac{1}{5}, -1$. 137. $1, -9$. 138. 2. 139. $3, 7$. 140. 12.

141. 8. 142. 10. 143. 3. 144. 1. 145. 80.

146. $-b$. 147. $\pm 2ab$. 148. $\frac{3a}{5}, \frac{4a}{5}$. 149. $a, \frac{-1 \pm \sqrt{-3}}{2} a$.

150. $\pm \frac{a}{\sqrt{2}}$. 151. $5, 6; \frac{1}{6}, -\frac{1}{3}; -\frac{19}{20}, \frac{3}{20}; 0, -\frac{9}{4}$.

152. Possible and unequal; possible and unequal; equal; impossible.

154. $65, \frac{1082}{16}$. 155. $21, \frac{1154}{16}$. 156. $9, -19$. 157. $117, \frac{655}{8}$.

158. $x^2-(p+q)x+pq=0$. 159. $qx^2-px+1=0$.

160. $x^2-(p^2-q)x+q(p^2-2q)=0$. 161. $x^2-(p^2-2q)x+q^2=0$.

162. $x^2-p(p^2-3q)x+q^3=0$.

163. $x^2-(p^3+p^2-3pq-2q)x+p(p^2-2q)(p^2-3q)=0$.

164. $x^2-(p^4-4p^2q+2q^2)x-q^4=0$. 173. $36, 21, 15$.

181. $5, 7; 7, 5$. 182. $11, 4; -4, -11$. 183. $116, -8; -12, -20$.

184. $13, 10; -\frac{17}{2}, -\frac{23}{2}$. 185. $\pm 12, \pm 7; \pm 7, \pm 12$.

186. $\pm 1, \pm 1; \pm \sqrt{3}, \mp \sqrt{3}$. 187. $\pm 2, \pm 3; \pm \frac{57}{\sqrt{1234}}, \mp \frac{35}{\sqrt{1234}}$.

188. $\pm 12, \pm 2; \pm \frac{4}{\sqrt{13}}, \mp \frac{38}{\sqrt{13}}$. 189. $15, 10; -\frac{40}{3}, -\frac{45}{4}$.

190. $8, 5; \frac{71}{6}, \frac{29}{12}$. 191. $\pm 11, \pm 2; \pm \frac{4}{\sqrt{6}}, \mp \frac{5}{2\sqrt{6}}$.

192. $\pm 4, \pm 1; \pm \frac{5}{\sqrt{2}}, \pm \frac{3}{\sqrt{2}}$. 193. $2, 3; 3, 2; \frac{-3 \pm \sqrt{17}}{2}, \frac{-3 \mp \sqrt{17}}{2}$.

194. $7, 8; -\frac{376}{7}, -\frac{476}{7}$. 195. $5, 3; 2, 6; -5, -1; -2, -4$.

196. $3, 2; -\frac{3}{5}, -\frac{2}{5}$. 197. $2, 2; \frac{2}{5}, -\frac{2}{3}$. 198. $0, 0; 12, 8$.

199. $8, 5; -\frac{14}{3}, -\frac{3}{7}$. 200. $\pm 4, \pm 3$. 201. $\pm 9, \pm 5$.

202. $6, 3; 2, -1$. 203. $\pm 5, \pm 3$. 204. $2, 1; -\frac{14}{19}, \frac{7}{19}$.

205. $10, 3; -3, -10$. 206. $7, -3; -3, 7$. 207. $1, -5; 5, -1$.

208. $8, 11; 11, 8$. 209. $\pm 4, \pm 3; \pm 3, \pm 4$. 210. $\pm 7, \mp 5; \pm 5, \mp 7$.

211. $3, 12; 12, 3$. 212. $4, \frac{1}{3}; -3, -\frac{1}{4}$. 213. $\frac{1}{2}, \frac{1}{4}; \frac{1}{4}, \frac{1}{2}$.

214. $\frac{1}{3}, -1; -\frac{2761}{10443}, \frac{2761}{5669}$. 215. $a, 0; 0, b$. 216. a, b.

217. $a^2, b^2; ab, ab$. 218. $2a, b; -a, -2b$. 219. $\pm a, \pm b$.

220. $\pm a, \pm (a-b)$. 221. $a, \frac{a+b}{2}; -\frac{a+b}{2}, -a$.

222. $a+b$, $a-b$; $a-b$, $a+b$. 223. $\pm\frac{a+b}{2}$, $\pm\frac{a-b}{2}$.

224. $4a$, $3b$; $-164a$, $-\frac{1}{4}\frac{2}{3}\frac{3}{3}b$.

225. $a(a+b+c)$, $b(a+b-c)$; $-a(a+b-c)$, $-b(a+b+c)$.

CHAPTER XIV.

1. 13. 2. 17. 3. 196. 4. 625. 5. 12. 6. 18.
7. 40. 8. 14. 9. 10, 10. 10. 6, 12. 11. 5, 16.
12. 4, 12. 13. 41. 14. 12, 17. 15. 9, 14. 16. 9, 11.
17. 9, 12. 18. 6, 8. 19. 3, 9. 20. 17, 18. 21. 11, 13.
22. 6, 7. 23. 8, 10. 24. 3, 4, 5. 25. 3, 6, 12. 26. 2, 5, 8.
27. 78. 28. 42. 29. 42. 30. 3. 31. $\frac{1}{4}$, $\frac{3}{4}$. 32. $\frac{4}{5}$, $\frac{5}{8}$.
33. $\frac{1}{3}$, $\frac{2}{3}$. 34. $\frac{1}{2}$, $\frac{1}{6}$. 35. Hypotenuse 30 in., sides 18, 24, in.
36. 5, 12, 13, ft. 37. 2, 16, in.; or 4, 18, in. 38. 110, 44, yds.
39. 14·832 in. 40. 24. 41. 48, 5. 42. 20. 43. 4 annas.
44. $1\frac{1}{2}d$. 45. 30. 46. 12. 47. 120. 48. 9.
49. 10, 12, days. 50. 30, 40, min. 51. 3, $3\frac{1}{2}$, miles an hour.
52. 10, 12 miles an hour. 53. Rs. 54, Rs. 30. 54. Re. 1. 6 as., 14 annas.
55. £50. 56. 3. 57. 12. 58. 10 dozen. 59. 36, 8.
60. 24, 3. 61. 28, 6, 2. 62. £500 at 5 p.c., £500 at 4 p.c.
63. Rs. 8000 at 4 p.c., Rs. 16000 at 3 p.c. 64. 1*s.* 1*d.*
65. 1*s.* 8*d.*, 1*s.* 2*d.* 66. 1*s.* 3*d.*, 1*s.* $1\frac{1}{2}d$. 67. 1*s.* 3*d.*
68. 480 acres, 40 shillings; or 84 acres, 7 shillings. 69. 160 acres.
70. 4 miles an hour. 71. 40 yards per minute. 72. Rs. 33 per dozen.
73. 3, $2\frac{1}{2}$, miles an hour. 74. Length 40 ft., height 24 ft., breadth 20 ft.
75. 5, 15. 76. 13, 16. 77. 8, 12. 78. 9, 15. 79. 5, 20.
80. 12, 60. 81. 16, 20. 82. 8, 12. 83. 5, 7. 84. 2, 9.
85. 5, 12. 86. 13, 20. 87. 4, 6. 88. 3, 7. 89. 1, 2.
90. 47. 91. 32 or 98. 92. 132, 110, yds.; 121, 120, yds.
93. 63, 84, days. 94. 42, 56, hours.
95. 10 days, 6 annas; 8 days, 5 annas. 96. 440, 352, yards a minute.
97. 150, 360, 390, yds. 98. 3, $3\frac{3}{4}$, yds.
99. 30, 20, miles an hour; 1400 miles. 100. $1\frac{1}{3}$, 2, miles an hour.

CHAPTER XV.

1. 7 : 8, 8 : 9, 11 : 12, 17 : 18. 2. 5 : 8, 13 : 20, 27 : 40, 11 : 16, 7 : 10.
3. 2 : 9. 4. 1 : 7. 5. 16 : 81; 27 : 125. 6. 1 : 8. 7. 2 : 3.
8. 1 : 20. 9. 3 : 11. 10. 1 : 4. 11. 1 : 20. 12. 11. 13. 4.
14. 19. 15. 11. 16. 17. 17. 32. 18. 7. 19. 16.
20. 54. 21. 1. 22. 3 : 1. 23. 3 : 2. 24. 1 : 1 or 1 : 2.

25. 2 : 5 or 3 : 4. 26. 1 : 6. 27. 3. 28. 10. 29. 5.
30. 4. 31. 10, 16. 32. 15, 20. 33. 15, 27. 34. 27, 12.
35. 30, 55. 36. 12, 18. 37. 40, 65. 38. 18, 24. 39. 4, 6.
40. 7, 8. 41. 8, 12. 42. 60, 36. 43. 100, 25. 44. 6, 10.
45. 2, 6. 48. $\frac{(m-n)\,ab}{nb-ma}$. 51. 39. 52. 14. 53. 21.
54. 2. 55. 23. 56. 11, 0. 57. 8, 1. 58. 10.
59. 5. 60. 2. 61. $x : y = y : a-x$, $y^2 : x = a-x : 1$, &c.
62. $x-y : y = a : x+y$, $y : x+y = x-y : a$, &c.
63. $x+y : a = a^2 : x^2-xy+y^2$, $x : a-y = a^2+ay+y^2 : x^2$, &c.
64. $x-y : x = a^2 : x^2+xy+y^2$, $y : x-a = x(x+a) : y^2$, &c.
65. $x^2-ax+a^2 : y^2 = y^2 : x^2+ax+a^2$, $x^2 : y^2-a^2 = y^2+a^2 : x^2+a^2$, &c.
77. 3, 6, 12. 78. 2, 8, 32. 79. 4, 8, 16. 80. 1, 3, 9.
81. 12, 21, 28, 49. 82. 10, 18, 30, 54. 83. 15, 30, 21, 42.
84. 4, 10, 12, 30. 85. 10, 15, 14, 21. 86. 6 ft. $10\frac{1}{2}$ in.
87. 96, 180, 204, yds. 88. 300, 150, 100. 89. Rs. 1250, Rs. 200, Rs. 50.
90. $\frac{3}{4}$, $\frac{5}{6}$. 96. 2, 3, −1. 97. 1, 3, 4. 98. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.
99. $b-\frac{c}{2}, c-\frac{a}{2}, a-\frac{b}{2}$.

MISCELLANEOUS EXAMPLES.

1. $4\frac{13}{15}$. 2. $(a+b+c)^3$. 4. (1) $5\frac{5}{11}$; (2) 2. 5. In 22 years.
6. Rs. 240, 120. 7. $\frac{8}{5}$. 8. $\frac{1}{3}x^3+\frac{1}{6}x^2-\frac{7}{9}x+\frac{4}{3}$, $a^{m+1}-(2a+a^m)\,c^n+2c^{2n}$.
9. $a^3+3a^2+9a+27$. 10. $\frac{x^2}{2}+\frac{2x}{3}+\frac{2}{x^2}$. 11. $2a+3x$, $4x\,(2a+3x)$.
12. (1) $\frac{1}{6}$; (2) x; (3) $\frac{2+x+3x^2}{2\,(1-x^4)}$; (4) $x^{2n}+2$. 13. (1) 5; (2) $\frac{7}{4}$.
14. 2 miles an hour. 15. a.
16. $x^8+a^8-2ax\,(x^6+a^6)+2a^3x^3\,(x^2+a^2)-2a^4x^4$. 17. $a-2p$.
18. (1) $\frac{3}{2}$; (2) 1, 2. 19. $\frac{a^2}{y^2+2a^2}$. 20. $(x \pm y)^2$. 21. 2, 1, 0.
22. (1) $16\frac{4}{11}$ min. past 3; (2) $49\frac{1}{11}$ min. past 3. 23. $\frac{109}{210}$.
24. $a^3+b^3+c^3-3abc$. 25. a^6-b^6. 26. $x-1$.
27. (1) 12; (2) 3, 2. 28. $\frac{49}{2}$, $\frac{1}{2}$. 29. $\frac{720}{49}$, $\frac{720}{41}$, $\frac{720}{31}$, days.
30. 432. 31. $7\frac{1}{2}$. 32. $a-b$. 33. $(a^3-1)\,(a+1)\,(a+4)$.
34. (1) $\frac{a^2-x^2}{a}$; (2) $4x\sqrt{x^2-1}$. 36. (1) 11; (2) $\frac{5}{4}$; (3) $\frac{1}{6}$, $\frac{1}{7}$.
37. $1\frac{7}{8}$ hours. 38. 12. 39. $-\frac{47}{12}$. 40. $2b\,(x+y)$.
41. a^5-b^5. 42. a^2-x^2. 43. (1) 0; (2) $\frac{a+b^2x}{b+ax}$.

45. (1) $\frac{26}{5}$; (2) $\frac{m^2-n^2}{ma-nb}$, $\frac{m^2-n^2}{mb-na}$. 46. $\frac{27}{4}$. 47. 2, 8 days.

48. $3a$. 49. 9, 9. 50. $a^6-22a^4+60a^3-55a^2+12a+4$.

51. $(x-1)^2$. 52. $\frac{x^4}{a^4}+\frac{x^2}{a^2}+1$. 53. $\frac{1}{abc}$.

55. (1) 1; (2) 16; (3) $\frac{a^2-b^2}{6a}$. 56. 30 hours. 57. 72 miles.

58. 536. 59. $3a^2+4ab+b^2$. 61. $\frac{4}{3(x+1)}$.

62. (1) $2x^2-x+1$; (2) $x^2-\frac{x}{2}+\frac{2}{x}$. 63. $5s.$; $4d.$ 64. 80 miles.

65. (1) 6; (2) 60. 66. $x^3-xy^2+y^3$. 67. $\frac{2}{a-b}$.

69. (1) 4; (2) 3; (3) 6, 10. 70. $32\frac{8}{11}$ minutes past 3.

71. Back Bay Rs. 1500, Mazagon Rs. 500.

72. $3\frac{6}{13}$ miles from Bassein. 73. $y^4+11y^3+47y^2+93y+69$.

74. x^4-16a^4. 75. (1) $\frac{2x-3}{(2x+3)(x^2-1)}$, (2) 1.

76. (1) $\frac{4}{3}$; (2) $\frac{ac}{b}$; (3) $\frac{nc+bd}{na+mb}$, $\frac{mc-ad}{na+mb}$; (4) $3a$, $-2b$.

77. $2x^2-3ax+4a^2$. 78. Rs. 17. 8 as.

79. $2\frac{1}{2}$, 2, miles an hour; 5 miles. 80. $4x^2+2xy$.

81. $a^3-2a^2b+4ab^2-8b^3$. 82. $(a+2b-c-3d)(a-2b+c-3d)$.

83. $2a^2x(2a-3x)$. 84. $\frac{2b-a}{a+b}$. 85. $x^3-2x^2y-y^2$.

86. (1) 9; (2) $\frac{27}{8}$; (3) 2, -1. 87. 56, 203.

88. 40, 24, days; Rs. 6. 4 as., Rs. 13. 12 as.

89. $6(x+2y)(x+4y)(x-y)^3$. 90. $\frac{x+y}{z}$. 92. (1) 4; (2) 1, $\frac{5}{4}$.

93. 1000 men. 94. $13\frac{1}{2}$ miles; 9 miles an hour.

95. $\frac{x^2+a^2}{x}$. 96. $\frac{1}{a^4-x^4}$. 97. $x(2x+3)$.

98. $\frac{x(x^2+2x-3)}{x^2-7x+22}$. 99. $a^2-a+\frac{1}{4}$. 100. 1, -1, 2, -2.

101. 800. 102. $(y^2-4)^2$; $x^4-4x^2yz+7y^2z^2$. 104. x^2+2x+3.

105. (1) 4; (2) $\frac{2}{2}$; (3) 12; (4) $\frac{21}{2}$. 106. Rs. 3,500; Rs. 4,500; Rs. 7,200.

107. Rs. 1,620; Rs. 1,180; Rs. 1,040. 108. 400 inches.

109. $a+1+\frac{1}{a}$; $\frac{x}{y}-\frac{2y}{x}$. 110. $x+1$; $120xy(x^2-y^2)$.

111. $2x^2+3xy+y^2$. 112. (1) 41; (2) 20; (3) 11, 9. 113. 91.

114. $\frac{x^2+y^2}{x^2-y^2}$. 115. x^3-2x^2+3x-4; $2x+5y$.

116. (1) $\frac{1}{2}$; (2) 26, 27; (3) 4, $-\frac{3}{2}$, 5. 117. *A* goes 17 miles; 4 hours.

118. Rs. $\frac{ma+nb+pc}{a+b+c}$ per gallon. 119. 324. 120. $\frac{3a^2+b^2}{4a+3b}$.

121. $\frac{1}{2a^2-1}$. 122. $a-x+1$. 123. 4, 5, 6.

124. 600. 125. $\frac{5}{6}$. 126. 14 miles.

127. $(x^2+y^2)^2-(z^2+1)^2$; $81x^4+27x^3y+9x^2y^2+3xy^3+y^4$. 128. $\frac{a}{a-b}$.

129. $1+\frac{x}{y}$. 130. $x-2a$; $(x^2-4)(x^2-9)$.

131. $2x^2+\frac{3}{2}x+5$; $x+\frac{1}{x}$. 132. (1) $\frac{1}{64}$; (2) 4; (3) 1, 2, 3.

133. 17, 14, 27, 8, 33. 134. $27\frac{3}{11}$ minutes past 5.

135. 80; 9 seers. 136. 54; 0. 137. (1) 1; (2) 1. 138. $2y^2$.

139. $(x+2y)(x-3y)$; $(x-y)(x-2y)(x+2y)$. 141. $6(x^2-1)(x^2-4)$.

142. $2x^2-\frac{3}{x}+\frac{4}{x^4}$; $2x^3-x^2-3$. 143. 1; no. 144. 2 hours.

145. 1*s.* 4*d.* 146. $2m-5x-3y$. 147. 1.

148. $3x-2$; $x(x-1)(x+1)(x+3)(x-6)$. 149. 1.

150. $x^2-3xy+2y^2$. 151. $\frac{5}{7}$, $\frac{7}{3}$. 152. Rs. 25. 153. $a^2+b^2+c^2$.

154. $x-2y$. 155. $\frac{3x^3+9x^2+2x+2}{3x(x+1)(x+2)}$; 1. 156. $\frac{x}{y}+\frac{y}{x}-\frac{1}{2}$.

157. Rs. 8000. 158. $2(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4)$.

159. $\frac{2}{a^2-b^2}$. 160. $x-2$; $\frac{x^3+2x^2-11x+6}{2x^2+4x-7}$. 161. 1, 2, 3.

162. $54\frac{6}{11}$ minutes past 10. 163. Rs. 60, Rs. 80. 164. ax^4+bx^2+c.

165. Vide Ex. 158. 166. a^2-x^2; $(a-x)(a^4-x^4)$.

167. (1) 9; (2) 20; (3) 9, 20. 168. 23.

170. (1) $(2x+3)(3x-2)$; (2) $(x-4)(3x+2)$; (3) $(x-3y)(x+3y)(3x-y)(3x+y)$.

171. x^2-3x+4. 172. $x+\frac{2}{x}-4$. 173. (1) 7; (2) $a+b$.

174. 30, $7\frac{1}{2}$, 6, years. 175. Rs. 83. 5 as. 4 p.

177. $x^2+4y^2+9z^2+6yz+3zx-2xy$. 178. $(b+c)(c+a)(a+b)$.

179. 2. 180. (1) 4; (2) 0. 181. $10\frac{5}{13}$ hours. 182. 12, 35, 5, 75.

183. $16(x^4-x^2y^2+y^4)-8a^2(x^2+y^2)+a^4$. 185. $(2x-5)(5x+1)$.

186. $(b-c)^2+(c-a)^2+(a-b)^2$. 187. $(x-a)^2$. 188. 91.

189. $4\frac{7}{8}$ miles. 190. x^8-16y^8.

191. $(a+b)(a+5b)$, $(a+b)(a-b)^2$, $(a-b)(a+6b)$; $(a+b)(a-b)^2(a+5b)(a+6b)$.

192. (1) $\frac{x-3}{x-4}$; (2) $\frac{1}{a+2}$. 193. (1) 6; (2) $\frac{pr}{p^2-q^2}$, $-\frac{qr}{p^2-q^2}$.

194. £33. 15*s.* 196. $(x+1)(x+4)^2$; $2(x+4)$. 197. 1.

198. $(a+b+c+d)(a-b-c+d)(a+b-c-d)(-a+b-c+d)$.

199. x^2+x-3; $(x^2+x-3)^2(x^2-x+3)(x^3-x+3)$.

200. $\frac{x^3}{4}-\frac{x}{3}-\frac{1}{2}$; $2x^2-3x+1$. 201. (1) $2a$; (2) 3, 2. 202. $4\frac{1}{2}$ miles.

205. (1) 0; (2) 1. 207. $(x-3)(x-4)(x+5)$. 208. a^2+b^2.

209. (1) 8; (2) $(a+b)(a+c)$. 210. 640.

211. $(x^2-pxy-y^2)(x^2+pxy-y^2)$.

213. $1+a+a^2+a^3+a^4+a^5+a^6+a^7+a^8+a^9$. 214. -1.

215. $a+b$, $a-b$. 216. 253. 217. 76 lbs., 30 lbs.

218. $(x+y)(2x-3y)$. 219. a. 220. $\left(\frac{p}{q}\right)^{m+n}$.

222. (1) a; (2) 5. 224. 84, 63. 225. $\frac{7}{6}$.

226. (1) $(2x+9y^3)(4x^2-18xy^3+81y^6)$; (2) $(x^2+7)(x^3-5)$;
(3) $(x-1)(x+1)(x+3)(x+5)$.

228. $abc(a+b+c)(b+c-a)(c+a-b)(a+b-c)$. 229. $4x^2+16x+11$.

230. (1) $\frac{2}{b}$; (2) $\frac{x^2+y^2}{(x+y)^2}$. 231. (1) 1; (2) 21, 6; (3) 2, 2.

232. £6. 6s. 233. 0. 235. 1. 236. $(x-1)^2$; $(x-1)^3(x+1)$.

237. $\frac{x}{a}-1+\frac{a}{x}$. 238. (1) $\frac{b}{a}(a-b+c)$; (2) 1, $\frac{1}{2}$, $\frac{1}{3}$. 239. 9, 6, gallons.

240. $x^{mn}+y^{mn}-\{x^ny^{(n-1)m}+x^{(m-1)n}y^m\}$; $x^2+xy\sqrt{2}+y^2$. 243. 0.

245. $(b+c)(c+a)(a+b)$. 246. $16a^4$.

247. $ax+2b+\frac{3c}{x}$; $x-\frac{1}{x}-1$. 248. 1 mile an hour.

249. (1) $\frac{x-3y}{x+3y}$; (2) $x^4+5x^2y^2+y^4$.

250. $x+4$; $(x-1)(x-2)(x+3)(x+4)(x-5)(x+6)$. 251. 1.

252. 7. 253. £4680, £4720. 254. 3.

255. (1) $(x^2-ax-a^2)(x^2+ax-a^2)$; (2) $(x+z+1)(x^2+z^2+1-x-z-xz)$;
(3) $(x-3y)(x+8y+1)$.

257. $a^3-\frac{1}{a^3}-3\left(a-\frac{1}{a}\right)$; $1-2x+3x^2$. 259. $-\frac{4}{3}$. 260. 360.

261. x^2+x+1; $4(x-3)(x+3)(3x-2)(3x+2)$.

262. (1) $\frac{1}{a}+\frac{1}{b}$; (2) $\frac{(x+1)^2}{x+2}$. 263. (1) $\frac{2}{2}\frac{3}{2}$; (2) $\frac{9}{2}$; (3) 3, 2.

264. $\frac{3}{8}$. 265. $7\frac{1}{2}$, 6, miles an hour. 267. $(2n+3)x+2(n+1)y$.

268. 1. 269. x^2+x+1. 270. $3x^2-4x+1$.

271. $(x^3+y^3+z^3-3xyz)^2$. 272. (1) $\frac{6}{2}\frac{9}{0}a$; (2) $a(b+c)$. 273. 81, 18.

274. (1) $\frac{x(1-x)}{2+x}$; (2) $\frac{2(x^2+1)}{x(x^2-1)}$. 276. $x+3$; $xy(x^2-y^2)$.

277. $\frac{x}{y}-\frac{y}{x}-\frac{1}{2}$; $x-\frac{3}{x}-2$. 278. (1) $-\frac{5}{4}$; (2) $\frac{7}{2}$, $-\frac{5}{2}$. 279. 22 days.

280. (1) $(a^2-\sqrt{2}ab+2b^2)(a^2+\sqrt{2}ab+2b^2)$; (2) $-(b-c)(c-a)(a-b)(a+b+c)$.

282. $-7, -28.$ 283. $\frac{(x+2)(2x+1)}{(x+1)(2x-1)}$.

286. Given expression $= (x^n - a^n)^2 + (ax^{n-1} - a^{n-1}x)(x^n - a^n)$.

287. $x^3 - 4x^2 + 2x - 3$. 288. $2x^2 + x + 1$; $\frac{1}{2}$, -1. 290. 36.

291. $9a^2 - 6ab + 4b^2$. 292. $x - 3y$; $(x^2-1)(x^2+1)(2x^2+x+1)$.

294. $3, -1, -2$. 298. 1.

299. $\{4(n^2+3n+1)\}^2$, where $2n$ is the first integer.

301. a^2d^2; $1+x^2-x^4-x^6$. 302. (1) 2; (2) 15.

303. 14, 11, yards. 304. $8\frac{2}{3}$, $3\frac{1}{3}$. 305. Rs. 525, Rs. 300.

307. $(x^2 - xy + y^2)^2$.

308. $(x-a)(x+a)(x^2+a^2)(x^2-ax+a^2)(x^2+ax+a^2)(x^4-a^2x^2+a^4)$.

309. $\frac{b}{a}$. 310. $x+5$. 311. (1) $\frac{3}{25}a$; (2) $\frac{ab}{a+b-c}$.

312. $4\frac{4}{5}$, $3\frac{3}{7}$, 24.

313. (1) 0; (2) $-\frac{x+c}{(x-a)(x-b)}$; (3) $\frac{a^2+b^2}{a}$.

314. (1) 7; (2) $-\frac{5}{6}$. 315. £2. 15*s*. 316. $\frac{7}{8}$.

317. $7x^2 - 7xy + 5y^2$. 318. $\frac{1}{a^4 - x^4}$. 319. (1) 8; (2) 16; (3) 1, 3.

320. $31\frac{17}{23}$, $42\frac{16}{17}$, $104\frac{2}{7}$, days. 321. $\frac{x^2+y^2}{x^2-y^2}$.

322. $\frac{1}{a^2} - \frac{2b}{a^3} + \frac{3b^2}{a^4}$; $-\frac{4b^3}{a^3} - \frac{3b^4}{a^4}$. 323. (1) 5; (2) 4; (3) $\frac{3c(a-b)}{2b}$.

324. $\frac{2}{3}$. 326. $a^7 - a^6x + a^5x^2 - a^4x^3 + a^3x^4 - a^2x^5 + ax^6 - x^7$.

327. (1) 5; (2) $\frac{3}{4}$; (3) 2, 3. 328. 24 feet.

329. $2(x^2+y^2+z^2-yz-zx-xy)$.

330. $x\sqrt{x} + y\sqrt{y} + z\sqrt{z} - 3\sqrt{xyz}$; $x^6 + ax^5 - a^3x^3 + a^5x + a^6$.

331. $\frac{1}{(x-1)(x-2)(x-3)}$. 332. (1) 10; (2) 5, 11. 333. $(x+y)^2 - 1$.

334. $(u+x+y+z)(-u+x-y+z)(u+x-y-z)(u-x-y+z)$.

335. $x^2 + 2x + 3$; $\frac{x-5}{x+5}$. 336. (1) $x+y+z$; (2) $\frac{1}{xyz}$; (3) 0.

337. 2. 338. (1) $\frac{bc^2+ca^2+ab^2-(a+b+c)}{bc+ca+ab-1}$; (2) $4\frac{1}{2}$; (3) $\frac{4a^2c}{4b^2+c^2}$.

339. -3. 340. $3a^2 - ab + b^2$; $\frac{a^4x^4-1}{a^2}$. 341. $x+3$; $xy(x^2-y^2)$.

342. (1) $\frac{1}{x+y}$; (2) $\frac{a+4}{a+5}$. 343. $x^2 + 2x + 3$.

344. (1) 24; (2) $\frac{a+b}{2}$; (3) $\frac{2pr}{p^2-2pq-q^2}$. 345. 20 years.

346. $2(y^2z^2+z^2x^2+x^2y^2) - (x^4+y^4+z^4)$. 347. $yz+zx+xy+z^2$. 348. 1.

349. $x^2 + \frac{1}{x^2} - 2$. 351. (1) 1; (2) 7. 352. $2x+5$.

353. $\frac{x-1}{x+1}$. 354. $\frac{xy}{x^2+y^2}$. 357. (1) 12; (2) $\frac{a^2+b^2+c^2}{3}$.

358. (1) $\frac{bc_1-b_1c}{ab_1-a_1b}$, $\frac{a_1c-ac_1}{ab_1-a_1b}$; (2) 4, 5, 6. 359. x^3+4x-1.

360. $3\sqrt{21}$. 361. $x-1$; $(x^2-4)(x^2+x-2)(x^3-x+1)$.

362. $-\frac{a^4+a^2b^2+b^4}{ab(a-b)^2}$. 364. $a-b+c-d$.

365. (1) 6; (2) 4; (3) 12, 12, 12. 366. 23.

367. $x^2-xy+y^2+x+y+1$. 368. x^2+1; $(x+2)(2x-1)(3x+1)$.

369. (1) $\frac{4a^3x}{x^4-a^4}$; (2) $\frac{3}{a+c}$; (3) $\frac{x(3x+1)}{4x^2+2x-1}$. 370. $x^2-\frac{3}{2}x-\frac{2}{3}$.

371. (1) 18; (2) $\frac{a^2-b^2}{ma-nb}$, $\frac{a^2-b^2}{na-mb}$. 372. $9a^2-6b^2+4bc+6ca-3ab$.

373. $x^4+x^3y+x^2y^2+xy^3+y^4$; $\frac{a^2}{b^2}-1+\frac{b^2}{a^2}$. 375. (1) x^a; (2) $\frac{(x^2+2)(x^4+1)}{x}$.

376. x^4-y^4. 377. $\frac{2x^2+3x-5}{7x-5}$. 378. $2(x^2+ax+2b^2)$.

379. (1) $-\frac{1}{3}$; (2) $\frac{6}{7}a$; (3) $\frac{1}{15}$, 18.

380. At noon, 125 miles distant from A. 381. $2x^5-x^4y-\frac{1}{3}x^3y^2-\frac{35}{6}x^2y^3+y^5$.

382. $2x(x+2)(x^2+2)(x-2)^2$. 383. $\frac{2x+3}{5x-2}$.

384. (1) $\frac{2a}{a+b}$; (2) $\frac{x(1-x)}{2+x}$. 385. $x^2-1+\frac{1}{x^2}$.

386. (1) 3; (2) $2\frac{1}{6}$, $3\frac{5}{9}$. 387. x^2+2x-3. 388. $3x-11$.

389. $\frac{x^3}{x^3+a^3}$. 390. $a^2-b^2+c^2-d^2$. 391. (1) 7; (2) 8, 5.

392. Rs. 500, Rs. 400, Rs. 200.

393. (1) $\sqrt{5xy}$; (2) $x+y$; (3) $\frac{2x^3}{x^4-1}$. 394. $x-2a$; $3ax(x^2-a^2)$.

395. $x^2-\frac{x}{2}+\frac{2}{x}$. 396. (1) $-\frac{4}{5}$; (2) $\frac{1}{2}$; (3) $2\frac{1}{2}$, $1\frac{1}{2}$.

397. 5 miles. 398. (1) $\frac{1}{x^m}$; (2) $\frac{a^2+b^2}{a^2-b^2}$; (3) $a-b$.

399. $(1-4x^2)(1+2x-4x^2)(1+2x+4x^2)$. 400. $3x^2-\frac{1}{3}xy+3y^2$.

401. 33·9; -7. 402. $-a+b-c$; $\frac{x^3}{y^3}-\frac{y^3}{x^3}$.

403. $2x^2-x-2$. 404. $\frac{1}{1-x^2}$.

406. (1) $7\frac{1}{2}$; (2) $\frac{6}{7}$; (3) $\frac{2}{5}$; (4) $\frac{3}{14}$, $\frac{5}{7}$. 407. 2080 bundles.

408. (1) $a+3b+2d$; (2) $8x^3$; (3) $\frac{2a^4}{(a^2+b^2)^2}$. 409. $2x-1$.

410. $x^2+\frac{1}{x^2}+1$. 411. $2+\frac{b}{2}-c$. 412. (1) 56; (2) 7, 9.

413. 4 days. 414. x^2-x-19. 415. $\frac{4(x^4+2)}{x^8+x^4+1}$.

416. $(x-1)(x-3)$. 417. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}$; 10·049875.

418. (1) $4\frac{1}{2}$; (2) $1\frac{2}{3}$; (3) 1, 2, 3. 419. 40 days.

420. 100 lbs.; 2 cwt., 3 cwt. 421. $x+y+z+xyz$.

422. Zero. 423. 1. 424. $ax+by+cz$. 425. (1) 6; (2) -3.

426. $a^{3n}+x^{3n}$; $x^{2n-1}-y^{2n-1}$. 427. $x+\frac{1}{2}$. 428. 2.

429. (1) $\frac{m(m+2k)}{2(m+k)}$; (2) 3, 3; (3) 4, 10. 430. 1.

431. $x-a$; $(x-a)^2(x-3a)(3x-7a)$. 432. $\frac{x^7}{a^3}-x^4-a^4$.

433. (1) 11; (2) 9; (3) 3, 2. 434. 3, 8, miles an hour.

435. $\frac{1}{x(x-a)(x-b)}$.

436. (1) $(x+6a)(x-11a)$; (2) $2(1+a)(1+c)(a-c)$.

438. (1) 5; (2) $\frac{3}{10}$; (3) $\frac{c(b-c)}{a(b-a)}$, $\frac{c(c-a)}{b(b-a)}$.

439. 120, 130, seconds. 440. x^6-a^6.

441. (1) $(x+6)(x+7)$; (2) $(x-6)(x+7)$; (3) $(7x+8y)(49x^2-56xy+64y^2)$.

442. $x-8$; $(x-8)^2(x+9)(3x-10)(3x+10)$.

443. (1) 3; (2) 5; (3) 8, -15. 444. $2\frac{5}{6}$ miles from P.

445. $a-b+c$. 446. $2x-y$. 447. $\frac{a^3}{(a-x)(a^3+x^3)}$.

448. $\frac{a^4-10a^3b-6ab^3-b^4}{a^4+10a^3b+6ab^3-b^4}$. 449. (1) $10\frac{3}{8}$; (2) 10; (3) -28, 10, 9.

450. 54; 646. 451. $-pa+b+q$.

452. (1) $(x-a)\left(x-\frac{1}{a}\right)$; (2) $(x^2-ax+a^2)(x^2+ax+a^2)$.

453. $(x^4-1)(x^6-1)$. 454. $x^{2n}+2$. 455. $n(n-1)$.

456. $1-\frac{1}{2}x-\frac{5}{8}x^2-\frac{5}{16}x^3$. 457. $\frac{mp-nq}{pa}$, $\frac{mp-nq}{qa}$. 458. £52; 52s.

459. 4 for 3d.; 512. 460. (1) $\frac{(m-n)(m-p)}{2m-n-p}$; (2) $\frac{x+1}{(x-1)(2x-1)}$.

461. $(x^2-3)^2(9x^2-1)(9x^4-1)$. 462. $\frac{a}{a-b}+\frac{b}{a+b}$.

463. (1) $\frac{b^2+ac}{b^2+c^2}$; (2) 5; (3) $\frac{a}{3}$; (4) 10, 20, 5. 464. £800.

465. $27\frac{3}{11}$ minutes past 5. 466. $x^2+y^2+z^2+yz+zx-xy$.

467. $(x^2+12ax+31a^2)^2-(4a^2)^2$.

468. $(x^2-ax+a^2)(x^2+ax+a^2)(x^4-a^2x^2+a^4)$;

$(x^2-2a^2)(x^2+2a^2)(x^2-2ax+2a^2)(x^2+2ax+2a^2)$.

469. $a+b+c$. 470. (1) $3\frac{1}{2}$; (2) 2, 3; (3) 1, 3, 5. 471. 600.

473. $2(a+b)x$. 474. $2(x^2+ax-2a^2)$.

475. (1) $bc-ca+ab$; (2) $x^2+y^2+z^2+x(y+z)$.

476. (1) $\frac{14}{13}$; (2) $-2\frac{39}{73}$; (3) 4, 5. 477. 10, 7, yards. 478. Zero.

479. 1. 480. $\frac{3(x-3a)(x-4a)}{2(x+3a)(x+4a)}$. 481. $\frac{x^2}{2}-2x+\frac{a}{3}$.

482. 5. 483. x^3-x+1. 484. $3x-4+\frac{1}{2x}$.

485. (1) 5; (2) $\frac{a+b}{2}$; (3) $\frac{1}{5}$, $\frac{1}{7}$. 486. 1220. 487. $x^4+5ax^3+6a^3x+a^4$.

488. $4a^2-3ab+b^2$. 489. (1) 2; (2) $\frac{ab-cd}{a+b-c-d}$. 490. Rs. 22, Rs. 24.

491. $3(yz+zx+xy)$. 492. 94. 493. $\frac{2a-3b+c}{2a-3c}$.

494. (1) $8\frac{8}{11}$; (2) $a-2b$. 495. 12 minutes past 4. 496. $a+b+c$.

497. x^2-x+1. 498. $2x^3-3x^2+x-4$. 499. (1) 2; (2) $-a$; (3) 3, 4.

500. 484, 348. 501. Zero. 502. $x-\sqrt{xy}+y$. 503. 15.

504. 200 miles; $33\frac{1}{3}$ miles an hour.

505. 59; $3(a^2+b^2+c^2)-2(bc+ca+ab)$. 506. $1+x-\frac{x^2}{2}$.

507. $\frac{bc-ca-ab}{b^2c^2-c^2a^2-a^2b^2}abc$, $\frac{bc-ca+ab}{b^2c^2-c^2a^2-a^2b^2}abc$. 508. 48.

509. $1-\frac{x}{6}+\frac{7x^2}{36}+\frac{5x^3}{216}+\frac{73x^4}{1296}$.

510. (1) $(x-a)(x+a)(x^2-ax+a^2)(x^2+ax+a^2)$; (2) $(x-2)(x-7)$.

511. (1) $\frac{a^2-b^2}{4a-b}$; (2) 37, $21\frac{1}{2}$, $11\frac{1}{2}$. 512. $7\frac{1}{5}$ days. 513. $a+b$.

514. x^2-y+5y^2. 515. b. 516. Shem survived Abram 35 years.

517. $1\frac{21}{40}$. 518. $(x^2-a^2)^2$.

519. $3xy(2x^5+13x^4y-23x^3y^2-18x^2y^3-23xy^4-5y^5)$. 520. $(a-x)(a+x)^3$.

521. $\frac{4ab}{a^2-b^2}$; (1) 1, (2) 0. 522. $\frac{1}{2}a-\frac{1}{3}b$.

523. (1) 1; (2) $\frac{4abc}{bc+ca+ab}$; (3) abc. 524. 52.

525. 5 sovereigns, 2 shillings. 526. $(a+b)^3+(a+b)^2+a+b$.

527. $a^2x^2+3bx+c$; $a^4x^4+6a^2bx^3+(2a^2c+9b^2)x^2+6bcx+c^2$.

528. (1) 1; (2) 2. 529. 1, $\pm(2a-x^2)$, $(2a-x^2)^2$, $(2a-x^2)^4$; a^4.

530. $4mx+6ny+3pz$. 531. (1) $\frac{b}{c}+\frac{c}{a}+\frac{a}{b}$; (2) $\frac{3}{2}$, $\frac{4}{3}$.

532. 132, 108, miles; 6 hours. 533. 46 pheasants, 80 partridges.

534. $a^2+b^2+c^2$. 535. x^4+9x^2+81; $a^3+2a^2b+2ab^2+b^3$.

536. $(x+1)(x+2)$; $\frac{1}{a^2}-\frac{1}{ab}+\frac{1}{b^2}$. 537. (1) x^3-2x+7; (2) $\sqrt{a}+\sqrt{b}-\sqrt{c}$.

538. (1) 1; (2) $bc+ca+ab$; (3) $\frac{b_1c-bc_1}{ab_1-a_1b}$, $\frac{ac_1-a_1c}{ab_1-a_1b}$; (4) $\frac{2}{5}$, $\frac{1}{10}$; (5) $\frac{1}{2}$, 1, $\frac{1}{3}$.

539. Rs. 3640. 540. $\frac{5}{9}$. 541. (1) 2; (2) 0. 542. 1.

543. $\left(\frac{1}{a}-b\right)^5-\frac{b}{a}\left(\frac{1}{a}-b\right)^3$. 544. $(a-b)x-(a+b)y$.

545. $a-2b$. 546. (1) $\frac{x-3}{x+3}$; (2) $\frac{4a^3}{a^4-x^4}$; (3) 1. 547. $a+2\sqrt{b}-3\sqrt[3]{c}$.

548. (1) $\frac{\sqrt{a}}{3}$; (2) a, b; (3) $\frac{24}{5}$, $\frac{12}{7}$, $\frac{8}{5}$. 549. 13.

551. (1) $2x^2-3ax+2a^2$; (2) $4x^2-2xy+y^2$.

552. $x+2$; $(x-1)(2x-1)(2x+1)(3x-2)$.

553. (1) $\frac{3a^2+2b^2}{5a(2a+3b)}$; (2) $\frac{2}{x^2}\sqrt{a^2-x^2}$. 554. $\frac{1}{a^2}+\frac{1}{ab}-\frac{1}{b^2}$.

555. (1) 10; (2) 1; (3) 2, 5, 10. 556. $\frac{am(2m+n)}{2(m^2-n^2)}$ miles.

557. x^4+9x^2+81.

558. (1) $(b-a)(a+b+c)(a+b-c)$; (2) $(a+b)(c-a+b)(c+a-b)$.

559. 1. 560. $(x+1)(x+3)(x-4)(x^2+4x+9)$.

562. (1) $\frac{x^2}{(x-a)^n}$; (2) $\frac{2c}{a-c}$. 563. (1) $-\frac{3}{4}$; (2) 72; (3) $\frac{b+c}{2a}$, $\frac{a+c}{2b}$.

564. 32, 40, $53\frac{1}{3}$, days. 565. $2\frac{x^4+1}{x^6-1}$.

566. $(a+b)x^3+(b+c)x^2+c+a$. 567. x^2-x+1. 569. $a^2+\frac{1}{a^2}+a-\frac{1}{a}$.

570. (1) $\frac{6}{7}$; (2) 8, 4, 5. 572. $a^3+(m+n)a^2+2mna+1$.

573. $2(ac+bd)$. 574. x^2-x+1; $(x-1)^2(x+1)(x-3)(2x+5)(x^2+5)$.

576. (1) $\frac{3abc}{a+b}$; (2) $a+b$.

577. (1) 4; (2) $\frac{abc}{a+b+c}$; (3) $\frac{1}{a+b+c}$, $\frac{2}{a+b+c}$, $\frac{3}{a+b+c}$.

578. (1) $\frac{ab-(n+1)(ca-bc)}{b-(n+1)(a-b)}$ hours; (2) $\frac{(n+1)ca-nbc-ab}{(n+1)(a-b)}$ hours.

579. $b-c$.

580. (1) $(x^2-x+1)(x^2+x+1)$; (2) $(b-c)(c-a)(a-b)(a+b+c)$.

582. $x-1$. 583. (1) $\frac{1}{x^2-x+1}$; (2) 4.

585. (1) $\frac{1}{9}$; (2) $\frac{a}{a+b}$, $\frac{b}{a+b}$; (3) 16, 36. 586. 18, 12, 15, feet.

587. $1-a^2$. 588. $x^8+ax^7-a^3x^5-a^4x^4-a^5x^3+a^7x+a^8$.

593. (1) 1; (2) $\frac{2}{(x+2)(x+3)}$. 594. $(b-c)(a-d)$.

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CONTENTS.

PAGE

GREEK AND LATIN CLASSICS:—

ANNOTATED AND CRITICAL EDITIONS 3
TEXTS 9
TRANSLATIONS 10
GRAMMAR AND COMPOSITION 15
HISTORY, GEOGRAPHY, AND REFERENCE BOOKS, ETC. . 18

MATHEMATICS:—

ARITHMETIC AND ALGEBRA 19
BOOKKEEPING 20
GEOMETRY AND EUCLID 21
ANALYTICAL GEOMETRY, ETC. 22
TRIGONOMETRY 22
MECHANICS AND NATURAL PHILOSOPHY . . . 22

MODERN LANGUAGES:—

ENGLISH 24
FRENCH CLASS BOOKS 29
FRENCH ANNOTATED EDITIONS 31
GERMAN CLASS BOOKS 32
GERMAN ANNOTATED EDITIONS 33
ITALIAN 34
BELL'S MODERN TRANSLATIONS 34

SCIENCE, TECHNOLOGY AND ART:—

CHEMISTRY 35
BOTANY 35
GEOLOGY 36
MEDICINE 36
BELL'S AGRICULTURAL SERIES 37
TECHNOLOGICAL HANDBOOKS 37
MUSIC 38
ART 39

MENTAL, MORAL AND SOCIAL SCIENCES:—

PSYCHOLOGY AND ETHICS 40
HISTORY OF PHILOSOPHY 41
LAW AND POLITICAL ECONOMY 41
HISTORY 42
DIVINITY, ETC. 44

SUMMARY OF SERIES 47

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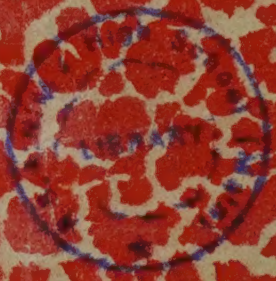
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